Sieve Methods in Combinatorics

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July 15, 2024

What are Sieve Methods?

Sieve methods in enumerative combinatorics are techniques for determining the cardinality of a set S by starting with a larger set and subtracting off or canceling unwanted elements. These methods can be broadly categorized into:

1. Methods approximating the answer with an overcount, then correcting the error iteratively.
2. Methods where elements of a larger set are weighted to cancel out unwanted elements, leaving the original set S.

Principle of Inclusion-Exclusion

Theorem 1

Let S be an n-set. Let V be the 2^n-dimensional vector space (over some field K) of all functions $f : 2^S \to K$. Define a linear transformation $\phi : V \to V$ by:

$$\phi f(T) = \sum_{Y \supseteq T} f(Y) \text{ for all } T \subseteq S. \tag{0.1}$$

Then ϕ^{-1} exists and is given by:

$$\phi^{-1} f(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y) \text{ for all } T \subseteq S. \tag{0.2}$$

Combinatorial Interpretation of PIE

In a combinatorial context, S represents properties that elements of set A may have. For any subset T of S, let $f_=(T)$ be the number of objects in A having exactly the properties in T, and let $f_\geq(T)$ be the number of objects in A that have at least the properties in T.

$$f_\geq(T) = \sum_{Y \supseteq T} f_=(Y) \tag{0.3}$$

$$f_=(T) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} f_\geq(Y) \tag{0.4}$$

The number of objects having none of the properties in S:

$$f_=(\emptyset) = \sum_{Y \supseteq T} (-1)^{\# Y} f_\geq(Y) \tag{0.5}$$

Dual Formulation of PIE

The dual formulation of PIE interchanges intersection ($\cap$) and union ($\cup$) operations. If:

$$\tilde{\phi} f(T) = \sum_{Y \subseteq T} f(Y) \text{ for all } T \subseteq S,$$

then $\tilde{\phi}^{-1}$ is given by:

$$\tilde{\phi}^{-1} f(T) = \sum_{Y \subseteq T} (-1)^{\#(T-Y)} f(Y) \text{ for all } T \subseteq S.$$

Similarly, if $f_{\leq}(T)$ is the number of objects of A having at most the properties in T, then:

$$f_{\leq}(T) = \sum_{Y \subseteq T} f = (Y) \tag{0.6}$$

$$f = (T) = \sum_{Y \subseteq T} (-1)^{\#(T-Y)} f_{\leq}(Y) \tag{0.7}$$

Derangements

Problem: How many permutations $w \in S_n$ have no fixed points, i.e., $w(i) \neq i$ for all $i \in [n]$? Such a permutation is called a derangement.
Solution: Denote this number by $D(n)$. We start with initial values:

$$D(0) = 1, \quad D(1) = 0, \quad D(2) = 1, \quad D(3) = 2.$$

Using the Principle of Inclusion-Exclusion:

$$D(n) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} i!.$$

This can be rewritten as:

$$D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

Rook Polynomials

- The derangement problem involves permutations $w \in S_n$ where $w(i) \neq i$.
- This concept extends to permutations with restricted positions using rook polynomials.

Definition 2

$B \subseteq [n] \times [n]$ is a board.

Definition 3

$G(w) = \{(i, w(i)) : i \in [n]\}$ is the graph of w.

Definition 4

N_j is the number of permutations $w \in S_n$ such that $j = \#(B \cap G(w))$.

Rook Polynomial:

$$r_B(x) = \sum_k r_k x^k.$$

Rook Polynomials (cont.)

The following result establishes a relationship between N_j and r_k.

Theorem 5

$$N_n(x) = \sum_j N_j x^j = \sum_{k=0}^{n} r_k (n-k)!(x-1)^k.$$

Rook Polynomials (cont.)

Example 6 (Derangements Revisited)

Take $B = \{(1,1), (2,2), \ldots, (n,n)\}$. We want to compute $N_0 = D(n)$, the number of derangements. Clearly, $r_k = \binom{n}{k}$, so

$$N_n(x) = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(x-1)^k.$$

Setting $x = 0$ gives:

$$N_0 = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(-1)^k = (-1)^k n!.$$

Ferrers Boards

Definition 7

A Ferrers board of shape $(b_1, \ldots, b_m)$ is defined by the integers $0 \leq b_1 \leq \cdots \leq b_m$ and consists of the set $B = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq b_i\}$, where we use Cartesian coordinates with the square $(1, 1)$ located at the bottom left.

Theorem 8

Let $\sum r_k x^k$ *be the rook polynomial of the Ferrers board* B *with shape* $(b_1, \ldots, b_m)$*. Define* $s_i = b_i - i + 1$*. Then:*

$$\sum_{k=0}^{m} r_k (x)_m - k = \prod_{i=1}^{m} (x + s_i).$$

Unimodal Sequences & V-partitions

Unimodal Sequence: - A sequence $d_1 d_2 \ldots d_m$ is unimodal if:

1. $\sum d_i = n$
2. There exists a j such that $d_1 \leq d_2 \leq \ldots \leq d_j \geq d_{j+1} \geq \ldots \geq d_m$

Generating Function for U(q)

$$U(q) = \sum_{n \geq 0} u(n) q^n = q + 2q^2 + 4q^3 + 8q^4 + 15q^5 + \ldots$$

Lemma 9

$$U(q) = \sum_{k \geq 1} \frac{q^k}{[k-1]![k]!}$$

Definition 10

A V-partition of n is an N-array:

$$\begin{bmatrix} & a_1 & a_2 & \cdots \\ c & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}$$

such that all numbers are natural,

$$c + \sum a_i + \sum b_i = n, \quad c \geq a_1 \geq a_2 \geq \ldots, \quad \text{and} \quad c \geq b_1 \geq b_2 \geq \ldots$$

Lemma 11

$$V(q) = \sum_{k \geq 0} \frac{q^k}{[k]!^2}$$

Unimodal Sequences & V-partitions (cont.)

Double Partitions: - N-array:

$$\begin{bmatrix} a_1 & a_2 & \cdots & \\ b_1 & b_2 & \cdots & \end{bmatrix}$$

Generating function $D(q)$:

$$D(q) = \sum_{n \geq 0} d(n) q^n = \prod_{k \geq 1} (1 - q^k)^{-2}$$

Define a map $F_1 : D_n \to V_n$ by:

$$F_1 \begin{bmatrix} a_1 & a_2 & \cdots & \\ b_1 & b_2 & \cdots & \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, & \text{if } a_1 \geq b_1 \\ \begin{bmatrix} & a_1 & a_2 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, & \text{if } b_1 > a_1 \end{cases}$$

Unimodal Sequences & V-partitions (cont.)

Define a new map $F_2 : D_{n-1} \to V_n^1$ by:

$$F_2 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1+1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, & \text{if } a_1+1 \geq b_1 \\ \begin{bmatrix} & a_1+1 & a_2 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, & \text{if } b_1 > a_1+1 \end{cases}$$

We continue to define maps $F_i : D_{n-\binom{i}{2}} \to V_n^{i-1}$ until $\binom{i}{2} > n$, so we obtain the following formula:

$$v(n) = d(n) - d(n-1) + d(n-3) - d(n-6) + \ldots$$

where we set $d(m) = 0$ for $m < 0$.

Unimodal Sequences & V-partitions (cont.)

Lemma 12

$U(q) + V(q) = D(q) = \prod_{k\geq 1}(1-q^k)^{-2}$

Theorem 13

$U(q) = \sum_{n\geq 1}(-1)^{n-1}q^{\binom{n+1}{2}}\prod_{k\geq 1}(1-q^k)^{-2}.$