SIEVE METHODS IN COMBINATORICS

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1. Introduction

In this paper, we explore various applications of sieve methods in combinatorial problems. Sieve methods in enumerative combinatorics are techniques for determining the cardinality of a set S by starting with a larger set and subtracting or canceling unwanted elements. These methods can be broadly categorized into two main approaches:

(1) Methods approximating the answer with an overcount, then correcting the error iteratively.
(2) Methods where elements of a larger set are weighted to cancel out unwanted elements, leaving the original set S.

Well known even for young mathematicians, the Principle of Inclusion-Exclusion is one of the most fundamental sieve methods in enumerative combinatorics. One of the simplest examples of using the Principle of Inclusion-Exclusion is:

Example 1.0.1. How many integers $1 \leq n \leq 1000$ are not divisible by 3, 5, or 7?

Proof. First, start from 1000. Then subtract separately the number of integers that are divisible by 3, 5, or 7. We get:

$$1000 - \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{7} \right\rfloor$$

We have subtracted more than once the numbers that are divisible by at least two of these three integers. So, we add back the numbers that are divisible by 15, 21, or 35:

$$1000 - \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{21} \right\rfloor + \left\lfloor \frac{1000}{35} \right\rfloor$$

Finally, we overcounted the numbers that are divisible by all three integers, so we subtract those:

$$1000 - \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{21} \right\rfloor + \left\lfloor \frac{1000}{35} \right\rfloor - \left\lfloor \frac{1000}{105} \right\rfloor = 457$$

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This simple example demonstrates how the Principle of Inclusion-Exclusion can be used to count objects by correcting overcounts. It turns out that the Principle of Inclusion-Exclusion applies in much more complex combinatorial problems.

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2. Inclusion-Exclusion

Abstractly, the Principle of Inclusion-Exclusion is nothing more than a small result in linear algebra, simply computing the inverse of a matrix. However, the importance of the Principle of Inclusion-Exclusion lies in its wide applicability across various problems. In this section, we will look at several problems that can be solved by the Principle of Inclusion-Exclusion. But first, we must look at the principle in its purest form.

2.1. Theorem (Principle of Inclusion-Exclusion).

Theorem 2.1. *Let S be an n-set. Let V be the 2^n-dimensional vector space (over some field K) of all functions $f : 2^S \to K$. Define a linear transformation $\phi : V \to V$ by:*

$$\phi f(T) = \sum_{Y \supseteq T} f(Y) \text{ for all } T \subseteq S. \tag{2.1}$$

Then ϕ^{-1} exists and is given by:

$$\phi^{-1} f(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y) \text{ for all } T \subseteq S. \tag{2.2}$$

Proof. We can start by defining $\psi : V \to V$ by:

$$\psi f(T) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} f(Y).$$

By composing functions right to left:

$$\phi\psi f(T) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} \phi f(Y) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} \sum_{Z \supseteq Y} f(Z).$$

Changing the order of summation:

$$= \sum_{Z \supseteq T} \left(\sum_{Z \supseteq Y \supseteq T} (-1)^{\#(Y-T)} \right) f(Z).$$

Setting $m = \#(Z - T)$, we have:

$$\sum_{Z \supseteq Y \supseteq T} (Z, T \text{ fixed}) (-1)^{\#(Y-T)} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} = \delta_{0m}.$$

Thus, $\phi\psi f(T) = f(T)$, implying $\psi = \phi^{-1}$. ∎

This is a linear algebraic representation of the Principle of Inclusion-Exclusion, defining a function over the power set of S and establishing a relationship between these subsets. While this theorem establishes the relationships between subsets of a given set S through linear transformations of summations and inversions, it can also be interpreted in a more combinatorial context.

2.2. Forms of the Principle of Inclusion-Exclusion. While the previous theorem establishes the relationships between subsets of a given set S through linear transformations of summations and inversions, the Principle of Inclusion-Exclusion can also be interpreted in other different contexts and forms.

2.2.1. *Combinatorial Interpretation.* In a typical combinatorial situation involving this theorem, S represents a set of properties that the elements of a given set A of objects may or may not have. For any subset T of S, let $f_=(T)$ be the number of objects in A that have exactly the properties in T (and fail to have the properties in $S - T$). More generally, if $w : A \to K$ is any weight function on A with values in a field (or abelian group) K, set $f_=(T) = \sum_x w(x)$ where x ranges over all objects in A having exactly the properties in T. Let $f_\geq(T)$ be the number of objects in A that have at least the properties in T.

Clearly:

$$f_\geq(T) = \sum_{Y \supseteq T} f_=(Y). \tag{2.3}$$

Hence by Theorem 2.1:

$$f_=(T) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} f_\geq(Y). \tag{2.4}$$

In particular, the number of objects having none of the properties in S is given by:

$$f_=(\emptyset) = \sum_{Y \supseteq T} (-1)^{\#Y} f_\geq(Y), \tag{2.5}$$

where Y ranges over all subsets of S. In typical applications of the Principle of Inclusion-Exclusion, it is relatively easy to compute $f_\geq(Y)$ for $Y \subseteq S$, so equation (2.4) yields a formula for $f_=(T)$.

In equation (2.4), we can think of $f_\geq(T)$ (the term indexed by $Y = T$) as being a first approximation to $f_=(T)$. Then, we can subtract:

$$\sum_{Y \supseteq T, \#(Y-T)=1} f_\geq(Y),$$

to get a better approximation. After, we add back in:

$$\sum_{Y \supseteq T, \#(Y-T)=2} f_\geq(Y),$$

and so on. We continue adding and subtracting, or "including" and "excluding" until we reach the explicit formula (2.4). This reasoning explains why this principle is called "Inclusion-Exclusion."

2.2.2. *Standard Formulation.* A more standard formulation of the Principle of Inclusion-Exclusion focuses on subsets of a finite set A rather than a set S of properties. We can let $A_1, A_2, \cdots, A_n$ be subsets of a finite set A. For each subset T of $[n]$, let:

$$A_T = \bigcap_{i \in T} A_i$$

where $A_\emptyset = A$. For $0 \leq k \leq n$ define:

$$S_k = \sum_{\#T=k} \#A_T,$$

which represents the sum of the cardinalities (or more generally the weighted cardinalities) of all k-tuple intersections of the A_i's. Then the number $\#(A_1 \cap \cdots \cap A_n)$ of elements of A that are in none of the A_i's is given by:

$$\#(A_1 \cap \cdots \cap A_n) = S_0 - S_1 + S_2 - \cdots + (-1)^n S_n,$$

where $S_0 = \#A$.

Writing this out more explicity, we get

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

which is a more familiar and common form of the Principle of Inclusion-Exclusion. It effectively counts the number of elements in the union of several sets by systematically including and excluding the sizes of various intersections. This completes the connection between the pure linear algebraic form of the Principle of Inclusion-Exclusion and its practical combinatorial application.

2.2.3. *Dual Formulation.* Now that we have established the Principle of Inclusion-Exclusion, we can extend to look at its dual formulation. THe dual formulation of the Principle of Inclusion-Exclusion is a complementary perspective that we can get by swapping certain operations and relationships in the original principle. Specifically,m we can dualize the principle of Inclusion-Exclusion by interchanging the intersection ($\cap$) and union ($\cup$) operations, as well as the subset ($\subseteq$) and superset ($\supseteq$)relations. The dual form of Theorem 2.1.1 states that if:

$$\tilde{\phi} f(T) = \sum_{Y \subseteq T} f(Y) \text{ for all } T \subseteq S,$$

then $\tilde{\phi}^{-1}$ exists and is given by:

$$\tilde{\phi}^{-1} f(T) = \sum_{Y \subseteq T} (-1)^{\#(T-Y)} f(Y) \text{ for all } T \subseteq S.$$

Similarly, if we let $f_{\le}(T)$ be the (weighted) number of objects of A having at most the properties in T, then:

$$f_{\le}(T) = \sum_{Y \subseteq T} f_{=}(Y)$$

and:

$$f_{=}(T) = \sum_{Y \subseteq T} (-1)^{\#(T-Y)} f_{\le}(Y).$$

What this dual formulation does is provide an alternative method for counting and analyzing sets, essentially flipping the perspective; Instead of including and excluding intersections of sets to count the elements in a union like the original principle, in the dual principle we include and exclude unions of sets to count the elements in an intersection. In combinatorial terms, this means that we count the elements that have at most the properties in T by including and excluding elements based on smaller sets that are contained within T.

2.2.4. *Common Special Case.* So far, we have explored the Principle of Inclusion-Exclusion in its original and dual formulations, demonstrating its variety of uses. A particularly useful scenario is when the function $f_=$ satisfies $f_=(T) = f_=(T')$ whenever $|T| = |T'|$, leading to a special case. In this case, the values $f_\geq(T)$ also depend only on $|T|$, leading to the equivalence of the following formulas based on (2.3) and (2.4):

$$b(m) = \sum_{i=0}^{m} \binom{m}{i} a(i), \quad 0 \leq m \leq n,$$

and:

$$a(m) = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} b(i), \quad 0 \leq m \leq n.$$

This matrix formulation provides a concrete way of understanding the Principle of Inclusion-Exclusion and its dual form, simplifying calculations through binomial coefficients.

2.3. **Example Applications.** Now that we know about the different forms of the Principle of Inclusion-Exclusion, we can see how it can be used in several examples. The first of the applications and one of the most common is the derangement problem.

2.3.1. *The Derangement Problem.* How many permutations $w \in S_n$ have no fixed points, i.e., $w(i) \neq i$ for all $i \in [n]$? Such a permutation is called a derangement.

To solve this, we can denote this number by $D(n)$. The initial values are $D(0) = 1$, $D(1) = 0$, $D(2) = 1$, $D(3) = 2$. Think of the condition $w(i) \neq i$ as the i-th property of w. The number of permutations with at least the set $T \subseteq [n]$ of points fixed is $f_\geq(T) = b(n-i) = (n-i)!$, where $\#T = i$ (since we fix the elements of T and permute the remaining $n-i$ elements arbitrarily).

Now, we can apply the Principle of Inclusion Exclusion to systematically count permutations by including and excluding fixed points. Eventually, we get that the number $f_=(\emptyset) = a(n) = D(n)$ of permutations with no fixed points is:

$$D(n) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} i!.$$

This last expression can be rewritten as:

$$D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).$$

Since:

$$e^{-1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \approx 0.36787944,$$

it is clear from the above formula that $n!/e$ is a good approximation to $D(n)$, and indeed it is not difficult to show that $D(n)$ is the nearest integer to $n!/e$.

Now that we have found the number of permutations $D(n)$, we can easily derive several things, starting withthe probability of a permutation being a derangement. By applying the Principle of Inclusion-Exclusion in a similar way, we get that the probability of a permutation being a derangement is

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

which tends to $\frac{1}{e}$ instead of $\frac{n!}{e}$.

It also follows immediately that for $n \geq 1$, we have the following recurrence relations:

$$D(n) = nD(n-1) + (-1)^n,$$

and:

$$D(n) = (n-1)(D(n-1) + D(n-2)).$$

Furthermore, in terms of generating functions, we have:

$$\sum_{n=0}^{\infty} \frac{D(n)x^n}{n!} = \frac{e^{-x}}{1-x}.$$

Now we can consider the function $b(i) = i!$, which has a very special property—it depends only on i, not on n. Equivalently, the number of permutations $w \in S_n$ that have at most the set $T \subseteq [n]$ of points unfixed depends only on $\#T$, not on n. This means that the equation for $D(n)$ can be rewritten in the language of the calculus of finite differences as:

$$D(n) = \Delta^n x!|_{x=0},$$

which is abbreviated as $\Delta^n 0!$.

Since the number $b(i)$ of permutations in S_n that have at most some specified i-set of points unfixed depends only on i, we can see that the same is true of the number $a(i)$ of permutations in S_n that have exactly some specified i-set of points unfixed. Thus, it is clear combinatorially that $a(i) = D(i)$, and this fact is also evident from the equations above. We can formally state the general result that follows from these considerations.

Proposition 2.2. *For each $n \in \mathbb{N}$, let B_n be a finite set, and let S_n be a set of n properties that elements of B_n may or may not have. Suppose that for every $T \subseteq S_n$, the number of $x \in B_n$ that lack at most the properties in T (i.e., that have at least the properties in $S_n - T$) depends only on $\#T$, not on n. Let $b(n) = \#B_n$, and let $a(n)$ be the number of objects $x \in B_n$ that have none of the properties in S_n. Then:*

$$a(n) = \Delta^n b(0).$$

2.3.2. *Example.* Now, let's consider an example where the previous proposition does not apply. Let $h(n)$ be the number of permutations of the multiset $M_n = \{1^2, 2^2, \ldots, n^2\}$ with no two consecutive terms equal. Thus $h(0) = 1$, $h(1) = 0$, and $h(2) = 2$ (corresponding to the permutations 1212 and 2121). Let P_i, for $1 \leq i \leq n$, be the property that the permutation w of M_n has two consecutive i's. Hence we seek $f_=(\emptyset) = h(n)$. It is clear by symmetry that for fixed n, $f_\geq(T)$ depends only on $i = \#T$, so write $g(i) = f_\geq(T)$. Clearly, $g(i)$ is equal to the number of permutations w of the multiset $\{1, 2, \ldots, i, (i+1)^2, \ldots, n^2\}$ (replace any $j \geq i$ appearing in w by two consecutive j's), so:

$$g(i) = (2n-i)!2^{-(n-i)}.$$

Note that:

$$b(i) := g(n-i) = (n+i)!2^{-i}$$

is not a function of i alone, so Proposition 2.2 is indeed inapplicable. However, we do get that:

$$h(n) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}(n+i)!2^{-i} = \Delta^n \left((n+i)!2^{-i}\right)_{i=0}.$$

Here the function $(n+i)!2^{-i}$ to which Δ^n is applied depends on n.

2.3.3. *Further Examples.* The descent set $D(w)$ of a permutation $w = a_1a_2\cdots a_n$ of $[n]$ by:

$$D(w) = \{i : a_i > a_{i+1}\}.$$

Our objective here is to obtain an expression for the quantity $\beta_n(S)$, the number of permutations $w \in S_n$ with descent set S. Let $\alpha_n(S)$ be the number of permutations whose descent set is contained in S. Thus:

$$\alpha_n(S) = \sum_{T\subseteq S} \beta_n(T).$$

It follows from the dual form of the principle that:

$$\beta_n(S) = \sum_{T\subseteq S} (-1)^{\#(S-T)}\alpha_n(T).$$

In addition, if the elements of S are given by $1 \le s_1 < s_2 < \cdots < s_k \le n-1$, then we have:

$$\alpha_n(S) = \binom{n}{s_1, s_2 - s_1, s_3 - s_2, \cdots n - s_k}.$$

Therefore:

$$\beta_n(S) = \sum_{1\le i_1<i_2<\cdots<i_j\le k} (-1)^{k-j}\binom{n}{s_{i_1}, s_{i_2} - s_{i_1}, \cdots n - s_{i_j}}.$$

We can write this in an alternative form as follows. Let f be any function defined on $[0, k+1]\times[0, k+1]$ satisfying $f(i, i) = 1$ and $f(i, j) = 0$ if $i > j$. Then the terms in the sum:

$$A_k = \sum_{1\le i_1<i_2<\cdots<i_j\le k} (-1)^{k-j} f(0, i_1)f(i_1, i_2)\cdots f(i_j, k+1)$$

are just the non-zero terms in the expansion of the $(k+1)\times(k+1)$ determinant with (i, j) entry $f(i, j+1)$, where $(i, j) \in [0, k]\times[0, k]$. Hence if we set $f(i, j) = \frac{1}{(s_j - s_i)!}$ (with $s_0 = 0, s_{k+1} = n$), we obtain from the equation for $\beta_n(S)$ that:

$$\beta_n(S) = n!\det\left[\frac{1}{(s_{j+1} - s_i)!}\right],$$

where $(i, j) \in [0, k]\times[0, k]$. For instance, if $n = 8$ and $S = \{1, 5\}$, then:

$$\beta_n(S) = 8!\begin{vmatrix} \frac{1}{1!} & \frac{1}{5!} & \frac{1}{8!} \\ 1 & \frac{1}{4!} & \frac{1}{7!} \\ 0 & 1 & \frac{1}{3!} \end{vmatrix} = 217.$$

By some manipulation this equation can also be written in the form:

$$\beta_n(S) = \det\left(\binom{n - s_i}{s_{j+1} - s_i}\right),$$

where $(i, j) \in [0, k]\times[0, k]$.

Analyzing why we obtained a determinant in this example, we get the following result.

Proposition 2.3. *Let $S = \{P_1, \ldots, P_n\}$ be a set of properties, and let $T = \{P_{s_1}, \ldots, P_{s_k}\} \subseteq S$, where $1 \le s_1 < \cdots < s_k \le n$. Suppose that $f_\le(T)$ has the form:*

$$f_\le(T) = h(n)e(s_0, s_1)e(s_1, s_2)\cdots e(s_k, s_{k+1}),$$

for certain functions h on $\mathbb{N}$ and e on $\mathbb{N} \times \mathbb{N}$, where we set $s_0 = 0, s_{k+1} = n, e(i,i) = 1$, and $e(i,j) = 0$ if $j < i$. Then:

$$f_=(T) = h(n) \det[e(s_i, s_{j+1})]_k^0.$$

This proposition generalizes the previous result by specifying conditions under which the function $f_\leq(T)$ can be represented in a form that yields a determinant for $f_=(T)$.

2.3.4. *Counting Surjections.* Having explored the Principle of Inclusion-Exclusion through the derangement problem, we now turn our attention to another classic use of the principle: counting surjections. Consider the following situation: There are m teachers and n children, where $m \geq N$. Each teacher gives one random child a cookie. What is the probability that all n children get at least one cookie? More formally, a function $f : [m] \to [n]$ is called a surjection if it covers all elements of $[n]$. There are n^m functions in total, and we are interested in how many of these are surjections.

Theorem 2.4. *The probability that all n children get cookies is:*

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \left(1 - \frac{k}{n}\right)^m.$$

Proof. We denote by A_i the set of functions that leave element i uncovered, i.e.,

$$A_i = \{f : [m] \to [n] \mid \forall j, f(j) \neq i\}.$$

The number of such functions is $(n-1)^m$, since we have $n-1$ choices for each of $f(1), f(2), \ldots, f(m)$. Similarly,

$$|A_I| = (n - |I|)^m \quad \text{for} \quad I \subseteq \{1, 2, \ldots, n\}$$

because we have $|I|$ forbidden choices for each function value.

By the Principle of Inclusion-Exclusion, we get that the number of functions which are not surjections is:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\emptyset \subseteq I \subseteq [n]} (-1)^{|I|+1} (n - |I|)^m.$$

Next, taking the complement, the number of surjections is:

$$n^m - \left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

To get the probability, we can divide by n^m to get:

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \left(1 - \frac{k}{n}\right)^m.$$

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In this proof, we utilized the Principle of Inclusion-Exclusion to calculate the number of surjections as well as the probability of a function being a surjection.

3. Further Extensions and Applications in Combinatorics

In the previous section, we explored the foundations of the Principle of Inclusion-Exclusion as a fundamental sieve method which helps us to count and estimate complex structures by systematically including and excluding overlapping cases. Having laid the groundwork, we now turn our attention to more advanced concepts including:

- Rook polynomials and their role in counting placements on chessboards
- Ferrers boards and their applications in enumerating partitions
- Unimodal sequences and V partitions

These topics represent deeper and more intricate applications of sieve methods in combinatorics.

3.1. **Rook Polynomials & Permutations with Restricted Positions.** The derangement problem previously mentioned involves finding the number of permutations $w \in S_n$ where certain values of $w(i)$ are disallowed for each i (specifically, $w(i) \neq i$). We can now extend this idea to a general theory of such permutations using the concept of rook polynomials, which is another application of sieve methods.

Let $B \subseteq [n] \times [n]$ be called a board. For a permutation $w \in S_n$, define the graph $G(w)$ of w by $G(w) = \{(i, w(i)) : i \in [n]\}$. Define N_j as the number of permutations $w \in S_n$ such that $j = \#(B \cap G(w))$. Let r_k be the number of k-subsets of B with no two elements sharing a coordinate, which is also the number of ways to place k non-attacking rooks on B.

We define the rook polynomial $r_B(x)$ of the board B by:

$$r_B(x) = \sum_k r_k x^k.$$

A permutation $w \in S_n$ can be identified with the placement of n non-attacking rooks on the squares $(i, w(i))$ of the board $[n] \times [n]$. Thus, N_j represents the number of ways to place n non-attacking rooks on $[n] \times [n]$ such that exactly j of these rooks are on B.

For instance, if $n = 4$ and $B = \{(1,1), (2,2), (3,3), (3,4), (4,4)\}$, then:

$$N_0 = 6, \quad N_1 = 9, \quad N_2 = 7, \quad N_3 = 1, \quad N_4 = 1,$$

and:

$$r_0 = 1, \quad r_1 = 5, \quad r_2 = 8, \quad r_3 = 5, \quad r_4 = 1.$$

Our goal is to express the numbers N_j, and particularly N_0, in terms of the numbers r_k. To this end, we define the polynomial $N_n(x)$ as follows:

Theorem 3.1. *We have:*

$$N_n(x) = \sum_j N_j x^j = \sum_{k=0}^{n} r_k (n-k)!(x-1)^k.$$

In particular,

$$N_0 = N_n(0) = \sum_{k=0}^{n} (-1)^k r_k (n-k)!.$$

We will present two semi-combinatorial proofs for this theorem: the first proof uses counting principles and algebraic manipulation, and the second proof constructs a bijection between the sets involved.

First Proof. Let C_k be the number of pairs (w, C), where $w \in S_n$ and C is a k-element subset of $B \cap G(w)$. For each j, choose w in N_j ways so that $j = \#(B \cap G(w))$, and then choose C in $\binom{j}{k}$ ways. Hence,

$$C_k = \sum_j \binom{j}{k} N_j.$$

On the other hand, we could first choose C in r_k ways and then "extend" to w in $(n-k)!$ ways. Therefore,

$$C_k = r_k(n-k)!.$$

Thus,

$$\sum_j \binom{j}{k} N_j = r_k(n-k)!.$$

Multiplying by y^k and summing over k,

$$\sum_j (y+1)^j N_j = \sum_k r_k(n-k)!y^k.$$

Setting $y = x - 1$ yields the desired formula.

Second Proof. Assume $x \in \mathbb{P}$. The left-hand side of the first equation of 3.1 counts the number of ways to place n non-attacking rooks on $[n] \times [n]$ and labeling each rook on B with an element of $[x]$. Alternatively, such a configuration can be obtained by placing k non-attacking rooks on B, labeling each with an element of $\{2, \ldots, x\}$, placing $n-k$ additional non-attacking rooks on $[n] \times [n]$ in $(n-k)!$ ways, and labeling the new rooks on B with 1. This argument establishes the desired bijection.

The given proofs are "semi-combinatorial" because they yield formulas involving parameters y and x, respectively, and we obtain 3.1 by setting $y = -1$ and $x = 0$.

As an example of Theorem 3.1, take $B = \{(1,1), (2,2), (3,3), (3,4), (4,4)\}$. Then,

$$N_4(x) = 4! + 5 \cdot 3!(x-1) + 8 \cdot 2!(x-1)^2 + 5 \cdot 1!(x-1)^3 + (x-1)^4 = x^4 + x^3 + 7x^2 + 9x + 6.$$

3.1.1. *Example 2.3.2 (Derangements Revisited).* In this example, we can revisit the derangement problem. We take $B = \{(1,1), (2,2), \ldots, (n,n)\}$. We want to compute $N_0 = D(n)$, the number of derangements. Clearly, $r_k = \binom{n}{k}$, so

$$N_n(x) = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(x-1)^k.$$

Setting $x = 0$ gives:

$$N_0 = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(-1)^k = (-1)^k n!.$$

3.1.2. *Example 2.3.3 (Mènage Problem).* Next, we consider the famous Mènage Problem. The problem asks for the number of ways of seating at a circular table with n married couples, husbands and wives alternating, so that no husband is next to his own wife. More formally, it asks for the number $M(n)$ of permutations $w \in S_n$ such that $w(i) \neq i, i+1 \mod n$ for all $i \in [n]$. In other words, we seek N_0 for the board $B = \{(1,1), (2,2), \ldots, (n,n), (1,2), (2,3), \ldots, (n-1,n), (n,1)\}$.

By examining the board B, we see that r_k is equal to the number of ways to choose k points, no two consecutive, from a collection of $2n$ points arranged in a circle.

3.1.3. *Lemma 2.3.4.* From these examples, we get that the number of ways to choose k points, no two consecutive, from a collection of m points arranged in a circle is given by the following lemma.

Lemma 3.2. *The number of ways to choose k points, no two consecutive, from a collection of m points arranged in a circle is:*

$$\binom{m}{m-k} = \binom{m-k}{k}.$$

We will now provide two proofs of this lemma: one that uses a general principle of converting circular arrangements to linear ones, and another that uses direct combinatorial arguments.

3.1.4. *First Proof.* Let $f(m,k)$ be the desired number, and let $g(m,k)$ be the number of ways to choose k nonconsecutive points from m points arranged in a circle, then coloring the k points red and one of the non-red points blue. Clearly, $g(m,k) = (m-k)f(m,k)$.

We can also compute $g(m,k)$ as follows. First, color a point blue in m ways. We now need to color k points red, no two consecutive, from a linear array of $m-1$ points. Place $m-1-k$ uncolored points on a line, and insert k red points into the $m-k$ spaces between the uncolored points (counting the beginning and end) in $\binom{m-k}{k}$ ways. Hence,

$$g(m,k) = m\binom{m-k}{k}, \text{ so } f(m,k) = \binom{m}{m-k}.$$

3.1.5. *Second Proof.* Label the points $1, 2, \ldots, m$ in clockwise order. We wish to color k of them red, no two consecutive. First, we count the number of ways when 1 isn't colored red. Place $m-k$ uncolored points on a circle, label one of these 1, and insert k red points into the $m-k$ spaces between the uncolored points in $\binom{m-k}{k}$ ways.

On the other hand, if 1 is to be colored red, then place $m-k-1$ points on the circle, color one of these points red and label it 1, and then insert in $\binom{m-k-1}{k-1}$ ways $k-1$ red points into the $m-k-1$ allowed spaces. Hence,

$$f(m,k) = \binom{m-k}{k} + \binom{m-k-1}{k-1} = \binom{m}{m-k}.$$

Now, we get the following corollary.

Corollary 3.3. *The polynomial $N_n(x)$ for the board B is given by:*

$$N_n(x) = \sum_{k=0}^{n} \binom{2n}{2n-k}(n-k)!(x-1)^k.$$

In particular, the number N_0 of permutations $w \in S_n$ such that $w(i) \neq i, i+1 \mod n$ for $1 \leq i \leq n$ is given by:

$$N_0 = \sum_{k=0}^{n} \binom{2n}{2n-k}(n-k)!(-1)^k.$$

Corollary 2.3.5 suggests an interesting extension. Let $1 \le k \le n$, and denote by $B_{n,k}$ the board $B_{n,k} = \{(i,i),(i,i+1),\dots,(i,i+k-1) \mod n : 1 \le i \le n\}$. We seek the rook polynomial $R_{n,k}(x) = \sum_i r_i(n,k)x^i$ of $B_{n,k}$. Thus, by equation (2.23), the number $f(n,k)$ of permutations $w \in S_n$ satisfying $w(i) \ne i, i+1, \dots, i+k-1 \mod n$ is given by:

$$f(n,k) = \sum_{i=0}^{n} (-1)^i r_i(n,k)(n-i)!.$$

Such permutations are termed k-discordant. For instance, 1-discordant permutations are simply derangements.

3.2. **Ferrers Boards.** When examining a particular board or class of boards B, it is interesting to investigate whether the rook numbers r_i exhibit any notable properties. In this context, we consider a specific class of boards known as Ferrers boards.

A Ferrers board of shape $(b_1,\dots,b_m)$ is defined by the integers $0 \le b_1 \le \cdots \le b_m$ and consists of the set $B = \{(i,j) : 1 \le i \le m, 1 \le j \le b_i\}$, where we use Cartesian coordinates with the square $(1,1)$ located at the bottom left. The board B essentially depends on the positive values of b_i. However, for technical reasons, we may allow $b_i = 0$. Notably, B can be viewed as a reflection and rotation of the Young diagram corresponding to the partition $\lambda = (b_m,\dots,b_1)$.

Theorem 3.4. *Let $\sum r_k x^k$ be the rook polynomial of the Ferrers board B with shape $(b_1,\dots,b_m)$. Define $s_i = b_i - i + 1$. Then:*

$$\sum_{k=0}^{m} r_k (x)_m - k = \prod_{i=1}^{m} (x + s_i).$$

To prove this, we will consider the Ferrers board extended by an additional rectangle.

Proof. Consider $x \in \mathbb{N}$, and let B' be the Ferrers board with shape $(b_1+x,\dots,b_m+x)$. View B' as the union of B and a rectangle C of size $x \times m$ placed below B.

We count $r_m(B')$ in two distinct ways: 1. Place k rooks on B in r_k ways and $m-k$ rooks on C in $(x)_{m-k}$ ways. This gives:

$$r_m(B') = \sum_{k=0}^{m} r_k (x)_{m-k}.$$

2. Place a rook in the first column of B' in $x + b_1 = x + s_1$ ways, a rook in the second column in $x + b_2 - 1 = x + s_2$ ways, and so forth, yielding:

$$r_m(B') = \prod_{i=1}^{m} (x + s_i).$$

This completes the proof. ■

With the theorem established, we can now explore specific cases and implications of this result.

Corollary 3.5. *Let B be the triangular (or staircase) board of shape $(0,1,2,\dots,m-1)$. Then $r_k = S(m, m-k)$.*

To demonstrate this, we will apply Theorem 3.4 to the specific case of the triangular board.

Proof. For this triangular board, each $s_i = 0$. Thus, by Theorem 3.4:

$$x^m = \sum_{k=0}^{m} r_k (x)_{m-k}.$$

which can be simplified to $r_k = S(m, m-k)$.

A combinatorial proof of this corollary is desirable. We can associate a partition of $[m]$ into $m-k$ blocks with the placement of k nonattacking rooks on $B = \{(i,j) : 1 \le i \le m, 1 \le j < i\}$. If a rook is at (i,j), then i and j are in the same block of the partition. This procedure yields the desired correspondence. ■

Moving forward, we extend our examination to the relationship between different Ferrers boards and their rook polynomials.

Corollary 3.6. *Two Ferrers boards, each with m columns (allowing empty columns), have the same rook polynomial if and only if their multisets of the numbers s_i are identical.*

Corollary 3.6 suggests investigating the number of Ferrers boards that have a rook polynomial equivalent to that of a given board B. This exploration provides deeper insights into the structure and properties of Ferrers boards, further demonstrating the applicability of rook polynomial analysis in combinatorial contexts.

In addition to the properties discussed, we can investigate the number of Ferrers boards that share the same rook polynomial. This leads us to the following theorem, which provides a formula for counting such Ferrers boards.

Theorem 3.7. *Let $0 \le c_1 \le \cdots \le c_m$, and let $f(c_1, \ldots, c_m)$ be the number of Ferrers boards with no empty columns and having the same rook polynomial as the Ferrers board of shape $(c_1, \ldots, c_m)$. Add enough initial 0's to $c_1, \ldots, c_m$ to get a shape $(b_1, \ldots, b_t) = (0, 0, \ldots, 0, c_1, \ldots, c_m)$ such that if $s_i = b_i - i + 1$, then $s_1 = 0$ and $s_i < 0$ for $2 \le i \le t$. Suppose that a_i of the s_j's are equal to $-i$, so $\sum_{i \ge 1} a_i = t - 1$. Then:*

$$f(c_1, \ldots, c_m) = \binom{a_1 + a_2 - 1}{a_2} \binom{a_2 + a_3 - 1}{a_3} \binom{a_3 + a_4 - 1}{a_4} \cdots .$$

To prove this, we will construct and count the required permutations of the multiset.

Proof. By Corollary 3.6, we seek the number of permutations $d_1 d_2 \ldots d_{t-1}$ of the multiset $\{1^{a_1}, 2^{a_2}, \ldots\}$ such that $0 \ge d_1 - 1 \ge d_2 - 2 \ge \cdots \ge d_{t-1} - t + 1$. Equivalently, $d_1 = 1$ and d_i must be followed by a number $d_{i+1} \le d_i + 1$.

Place the a_1 1's down in a line. The a_2 2's may be placed arbitrarily in the a_1 spaces following each 1 in $\binom{a_1+a_2-1}{a_2}$ ways. Now the a_3's may be placed arbitrarily in the a_2 spaces following each 2 in $\binom{a_2+a_3-1}{a_3}$ ways, and so on, completing the proof. ■

For instance, there are no other Ferrers boards with the same rook polynomial as the triangular board $(0, 1, \ldots, n-1)$, while there are 3^{n-1} Ferrers boards with the same rook polynomial as the $n \times n$ chessboard $[n] \times [n]$.

Having established the theorem, we now consider a scenario where all columns of the Ferrers board must have distinct lengths.

If in the proof of Theorem 3.7 we want all the columns of our Ferrers board to have distinct lengths, then we must arrange the multiset $\{1^{a_1}, 2^{a_2}, \ldots\}$ to first strictly increase from 1 to its maximum in unit steps and then to be non-increasing. Hence, we obtain the following result.

Corollary 3.8. *Let B be a Ferrers board. Then there is a unique Ferrers board whose columns have distinct (nonzero) lengths and that has the same rook polynomial as B.*

Proof. For instance, the unique "increasing" Ferrers board with the same rook polynomial as $[n] \times [n]$ has shape $(1, 3, 5, \ldots, 2n-1)$. ■

This result shows the connection between Ferrers boards and their corresponding rook polynomials. With this understanding, we can now move into applications of other sieve methods.

3.3. **Unimodal Sequences & V-Partitions.** We now present an example of a sieve process that cannot be easily derived using the Principle of Inclusion-Exclusion. This involves the concept of unimodal sequences, also known as n-stacks. A unimodal sequence of weight n is defined as a sequence $d_1 d_2 \ldots d_m$ such that:

(1) $\sum d_i = n$
(2) $\exists\ j$: $d_1 \leq d_2 \leq \ldots \leq d_j \geq d_{j+1} \geq \ldots \geq d_m$

We can find a generating function $U(q)$ of the total number of unimodal sequences of weight n. Let $u(n)$ denote the total number of unimodal sequences with weight n with $u(0) = 0$. For instance, $u(5) = 15$ because 5 has 16 compositions and all of them are unimodal except for 212. We can also introduce the corresponding generating function as follows:

$$U(q) = \sum_{n \geq 0} u(n) q^n = q + 2q^2 + 4q^3 + 8q^4 + 15q^5 + 27q^6 + 47q^7 + 79q^8 + \ldots$$

Our goal is to find an explicit formula for $U(q)$ and ultimately a product type formula. For this purpose, we write $[k]! = (1-q)(1-q^2)\cdots(1-q^k)$.

Lemma 3.9.

$$U(q) = \sum_{k \geq 1} \frac{q^k}{[k-1]![k]!}$$

Proof. It is clear that every unimodal sequence with the largest term k has the form

$$w = a_1 a_2 \cdots a_m = \underbrace{11\cdots1}_{b_1}\underbrace{22\cdots2}_{b_2}\cdots\underbrace{kk\cdots k}_{b_k}\underbrace{(k-1)\cdots(k-1)}_{c_{k-1}}\underbrace{(k-2)\cdots(k-2)}_{c_{k-2}}$$

for some $b_1, \ldots, b_{k-1}, c_1, \ldots, c_{k-1} \geq 0$ and $b_k \geq 1$.

Next, we can rewrite $\frac{q^k}{[k]![k-1]!}$. This represents the number of unimodal sequences where the largest term is k. We can write it in the form:

$$\frac{q^k}{[k]![k-1]!} = (1+q+q^2+\cdots)\cdots(1+q^{k-1}+q^{2(k-1)}+\cdots)(q^k+q^{2k}+q^{3k}+\cdots)(1+q^{k-1}+\cdots)\cdots(1+q+q^2+\cdots).$$

Having b_1 of 1's in w will cause us to choose $q \cdot b_1$ from the first bracket, having b_2 of 2's in w will cause us to choose $q^2 \cdot b_2$ from the second bracket, and so on. We note that the free term 1 is missing in $(q^k + q^{2k} + \cdots)$ due to the restriction $b_k \geq 1$. ■

3.3.1. *V-Partitions.* It turns out in obtaining a product formula for $U(q)$, it is more manageable to work with objects slightly different from unimodal sequences and then connect them to unimodal sequences. We define a a V-partition of n to be an N-array:

$$\begin{bmatrix} & a_1 & a_2 & \cdots \\ c & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}$$

such that all numbers are natural,

$$c + \sum a_i + \sum b_i = n, \quad c \geq a_1 \geq a_2 \geq \ldots, \quad \text{and} \quad c \geq b_1 \geq b_2 \geq \ldots$$

Hence, a V-partition may be regarded as a unimodal sequence of the same weight, but "rooted" at the largest element. Let $v(n)$ be the number of V-partitions of n, with $v(0) = 1$. For instance, $v(4) = 12$, since there is one way of rooting 4, one way for 13, one for 31, two for 22, one for 211, one for 112, and four for 1111.

Next, we define the following lemma.

Lemma 3.10.

$$V(q) = \sum_{n \geq 0} v(n) q^n = \sum_{k \geq 0} \frac{q^k}{[k]!^2}$$

Proof. Proving this is straightforward and analogous to the proof of Lemma 3.9. Just note that we presently have two brackets $(1 + q^k + q^{2k} + \cdots$,because we should root the largest element in the V-partition. ■

Now, let us define the set D_n of double partitions of n, that is, N-arrays:

$$\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix}$$

such that $a_i, b_j \in \mathbb{N}$, $\sum a_i + \sum b_i = n$, $a_1 \geq a_2 \geq \ldots$, and $b_1 \geq b_2 \geq \ldots$. If we set $d(n) = |D_n|$, then the following result is clear:

Lemma 3.11.

$$D(q) = \sum_{n \geq 0} d(n) q^n = \prod_{k \geq 1} (1 - q^k)^{-2}$$

Now, let V_n be the set of V-partitions of n, so that $|V_n| = v(n)$. We define a map $F_1 : D_n \to V_n$ by:

$$F_1 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, & \text{if } a_1 \geq b_1 \\ \begin{bmatrix} & a_1 & a_2 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, & \text{if } b_1 > a_1 \end{cases}$$

F_1 is surjective, but it is not injective. Every V-partition in the form $\begin{bmatrix} & a_1 & a_2 & \cdots \\ c & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}$ with $c > a_1$ appears twice, because it is the image of both $\begin{bmatrix} c & a_1 & a_2 & \cdots \\ b_1 & b_2 & b_3 & \cdots \end{bmatrix}$ and $\begin{bmatrix} a_1 & a_2 & a_3 & \cdots \\ c & b_2 & b_3 & \cdots \end{bmatrix}$. All other V-partitions are counted once. We can call the set of the former V- partitions, those with $c > a_1$, as V_n^1. Then, we get

$$\|V_n| = |D_n| - |V_n^1|$$

Next, we define a new map $F_2 : D_{n-1} \to V_n^1$ by:

$$F_2 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1+1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, & \text{if } a_1+1 \geq b_1 \\ \begin{bmatrix} & a_1+1 & a_2 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, & \text{if } b_1 > a_1+1 \end{cases}$$

Again, F_2 is subjective, but every V-partition that has a form $\begin{bmatrix} & a_1 & a_2 & \cdots \\ c & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}$ with $c > a_1 > a_2$ appears twice, because it arises as the image of both $\begin{bmatrix} a_1-1 & a_2 & a_3 & \cdots \\ c & b_1 & b_2 & \cdots \end{bmatrix}$ and $\begin{bmatrix} c-1 & a_1 & a_2 & \cdots \\ b_1 & b_2 & b_3 & \cdots \end{bmatrix}$. All other V-partitions in V_n^1 are counted exactly once. We can name the set of the former V-partitions as V_n^2. Then, we get

$$|V_n| = |D_n| - |V_n^1| = |D_n| - |D_{n-1}| + |V_n^2|$$

.

Next, we define a new map $F_3 : D_{n-3} \to V_n^2$ by:

$$F_3 \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2+1 & a_3 & \cdots \\ a_1+2 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, & \text{if } a_1+2 \geq b_1 \\ \begin{bmatrix} & a_1+2 & a_2+1 & \cdots \\ b_1 & & & \\ & b_2 & b_3 & \cdots \end{bmatrix}, & \text{if } b_1 > a_1+2 \end{cases}$$

Using the same logic as above, we denote a subset of V_n^2 with $c > a_1 > a_2 > a_3$ as V_n^3 and we get

$$|V_n| = |D_n| - |D_{n-1}| + |D_{n-3}| - |V_n^3|.$$

By continuing this process, we define maps $F_i : D_{n-\binom{i}{2}} \to V_n^{i-1}$ until $\binom{i}{2} > n$, so we obtain the following sieve-theoretic formula:

$$v(n) = d(n) - d(n-1) + d(n-3) - d(n-6) + \ldots$$

where we set $d(m) = 0$ for $m < 0$. Then, from Lemma 3.11, we obtain the following result:

$$V(q) = \sum_{n\geq 0} \left((-1)^n q^{\binom{n+1}{2}}\right) D(q) = \sum_{n\geq 0} \left((-1)^n q^{\binom{n+1}{2}}\right) \Pi_{k\geq 1}(1-q^k)^{-2}.$$

We also have the following simple result which connects all the generating functions introduced above:

Lemma 3.12. $U(q) + V(q) = D(q) = \Pi_{k\geq 1}(1-q^k)^{-2}$

Proof. Let U_n be the set of all unimodal sequences of weight n. We need to find a bijection $D_n \leftrightarrow U_n \cup V_n$. Such a bijection is given by:

$$\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = \begin{cases} \begin{bmatrix} & a_2 & a_3 & \cdots \\ a_1 & & & \\ & b_1 & b_2 & \cdots \end{bmatrix}, & \text{if } a_1 \geq b_1 \\ \cdots a_2\ a_1\ b_1\ b_2 \cdots, & \text{if } b_1 > a_1 \end{cases}$$

This bijection allows us to connect the structures of double partitions and V-partitions to unimodal sequences. ■

Now, having established the relationships between V-partitions, unimodal sequences, and their generating functions, we can derive the product type formula for $U(q)$ that we were originally looking for:

Theorem 3.13. $U(q) = \sum_{n\geq 1}(-1)^{n-1} q^{\binom{n+1}{2}} \prod_{k\geq 1}(1-q^k)^{-2}$.

This theorem follows directly from the previous lemmas, as we will see in the following proof.

Proof. Combining Lemma 3.12 as well as the equation for $V(q)$, we get:

$$U(q) = D(q) - V(q) = \Pi_{k\geq 1}(1-q^k)^{-2} - \sum_{n\geq 0} \left((-1)^n q^{\binom{n+1}{2}}\right) \Pi_{k\geq 1}(1-q^k)^{-2} =$$

$$\Pi_{k\geq 1}(1-q^k)^{-2} \sum_{n\geq 0} \left((-1)^{n-1} q^{\binom{n+1}{2}}\right)$$

■

Now that we have explored some more advanced sieving methods of the first type in unimodal sequences and V-partitions, we will turn our attention to the second type of sieve method with involutions and the Garsia-Milne sieve.

4. Advanced Sieve Methods

In this section, we explore a second type of sieve method that leverages the idea of weighted elements within a larger set. By carefully assigning weights, unwanted elements can effectively cancel each other out, leaving only the desired set S. This approach is exemplified through the use of involutions and the Garsia-Milne sieve, which provide powerful tools for solving complex combinatorial problems.

4.1. **Involutions.** In this subsection, we will explore the concept of involutions and their application in combinatorial proofs. But first, let us define an involution.

Definition 4.1. An involution is a function τ from a set X to itself such that applying τ twice returns every element to itself, i.e., $\tau(\tau(x)) = x$ for all $x \in X$.

Involutions are useful in proving combinatorial identities as they can establish bijections between sets, thus determining that they have the same cardinality. To show this, we consider the following combinatorial identity:

Lemma 4.2.

$$f_=(\emptyset) + \sum_{\#Y\ odd} f_\geq(Y) = \sum_{\#Y\ even} f_\geq(Y)$$

where $f_=(Y)$ (respectively, $f_\geq(Y)$) denotes the number of objects in a set A having exactly (respectively, at least) the properties in $T \subseteq S$.

This identity is essentially identity (2.5). However, unlike (2.5) what we are trying to show here is that two sets have the same cardinality. In the following proof, we will use an involution to establish a bijection between the two sets and therefore prove they have the same cardinality.

Proof. The left-hand side of this identity is the cardinality of the set $M \cup N$, where M is the set of objects x having none of the properties in S, and N is the set of ordered triples (x, Y, Z), where $x \in A$ has exactly the properties $Z \subseteq Y$ with $\#Y$ odd. The right-hand side of this identity is the cardinality of the set N' of ordered triples (x', Y', Z'), where $x' \in A$ has exactly the properties $Z' \subseteq Y'$ with $\#Y'$ even.

To prove this, we totally order the set S of properties, and define $\sigma : M \cup N \to N'$ as follows:

$$\sigma(x) = (x, \emptyset, \emptyset), \quad \text{if } x \in M$$

$$\sigma(x, Y, Z) = \begin{cases} (x, Y - i, Z), & \text{if } (x, Y, Z) \in N \text{ and } \min Y = \min Z = i \\ (x, Y \cup i, Z), & \text{if } (x, Y, Z) \in N \text{ and } \min Z = i < \min Y \end{cases}$$

It is easily seen that σ is a bijection with inverse

$$\sigma^{-1}(x, Y, Z) = \begin{cases} x \in M, & \text{if } Y = Z = \emptyset \\ (x, Y - i, Z) \in N, & \text{if } Y \neq \emptyset \text{ and } \min Y = \min Z = i \\ (x, Y \cup i, Z) \in N, & \text{if } Z \neq \emptyset \text{ and } \min Z = i < \min Y \end{cases}$$

(where we set $\min Y = \infty$ if $Y = \emptyset$).

This construction yields the desired bijective proof of the identity. ■

Note that if in the definition of σ^{-1} we identify $x \in M$ with $(x, \emptyset, \emptyset) \in N'$ (so $\sigma^{-1}(x, \emptyset, \emptyset) = (x, \emptyset, \emptyset)$), then $\sigma \cup \sigma^{-1}$ is a function $\tau : N \cup N' \to N \cup N'$ satisfying:

1. τ is an involution; that is, $\tau^2 = \text{id}$. 2. The fixed points of τ are the triples $(x, \emptyset, \emptyset)$, so they are in one-to-one correspondence with M. 3. If (x, Y, Z) is not a fixed point of τ and we set $\tau(x, Y, Z) = (x, Y', Z')$, then $(-1)^{\#Y} + (-1)^{\#Y'} = 0$.

Thus, the involution τ selects terms from the right-hand side of the identity in Lemma 4.2 (or rather, terms from the right-hand side after each $f_\geq(Y)$ is written as a sum) that add up to the left-hand side, and then τ cancels out the remaining terms, using a sieving process to establish the bijection.

4.1.1. *General Context.* To discuss involutions more generally, suppose that a finite set X is written as a disjoint union $X^+ \cup X^-$ of two subsets, called the "positive" and "negative" parts of X, respectively. Let τ be an involution on X that satisfies the following conditions:

(1) If $\tau(x) = y$ and $x \neq y$, then exactly one of x or y belongs to X^+ (so the other belongs to X^-).
(2) If $\tau(x) = x$, then $x \in X^+$.

If we define a weight function w on X by

$$w(x) = \begin{cases} 1, & \text{if } x \in X^+ \\ -1, & \text{if } x \in X^- \end{cases}$$

then clearly

$$\#\text{Fix}(\tau) = \sum_{x \in X} w(x),$$

where $\text{Fix}(\tau)$ denotes the fixed point set of τ. Just as in the previous paragraph, the involution τ selects terms from the right-hand side of this sum that add up to the left-hand side, and cancels the remaining terms.

4.1.2. *The Involution Principle.* Consider another set X' that is also expressed as a disjoint union $X' = X'^+ \cup X'^-$, and an involution $\tilde{\tau}$ on X' satisfying the conditions:

(1) If $\tilde{\tau}(x) = y$ and $x \neq y$, then exactly one of x or y belongs to X'^+ (so the other belongs to X'^-).
(2) If $\tilde{\tau}(x) = x$, then $x \in X'^+$.

Suppose we have a sign-preserving bijection $f : X \to X'$, meaning $f(X^+) = X'^+$ and $f(X^-) = X'^-$. Then $\#\text{Fix}(\tau) = \#\text{Fix}(\tilde{\tau})$ since $\#\text{Fix}(\tau) = \#X^+ - \#X^-$ and $\#\text{Fix}(\tilde{\tau}) = \#X'^+ - \#X'^-$. We wish to construct a bijection g between $\text{Fix}(\tau)$ and $\text{Fix}(\tilde{\tau})$. This construction is known as the *involution principle.* It is a powerful technique for turning non-combinatorial proofs into combinatorial ones and is defined as follows.

Theorem 4.3. *Let $x \in Fix(\tau)$. It is easily seen, since X is finite, that there is a nonnegative integer n for which*

$$f(\tau f^{-1} \tilde{\tau} f)^n(x) \in Fix(\tilde{\tau}).$$

Define $g(x)$ to be $f(\tau f^{-1} \tilde{\tau} f)^n(x)$ where n is the least nonnegative integer for which the above equation holds.

Thus, the involution τ pairs elements such that the contributions of the unwanted elements, which are the non-fixed points, cancel each other out, leaving only the contributions of the fixed points, which correspond to the elements of the set M.

Next, we provide a nice geometric way to visualize the situation better. Represent the elements of X and $\tilde{X}$ as vertices of a graph Γ. Draw an undirected edge between two distinct vertices x and y if (1) $x, y \in X$ and $\tau(x) = y$; or (2) $x, y \in \tilde{X}$ and $\tilde{\tau}(x) = y$; or (3) $x \in X, y \in \tilde{X}$, and $f(x) = y$. Every component of Γ will then be either a cycle disjoint from $\text{Fix}(\tau)$ and $\text{Fix}(\tilde{\tau})$, or a path with one endpoint z in $\text{Fix}(\tau)$ and the other endpoint $\tilde{z}$ in $\text{Fix}(\tilde{\tau})$. Then g is defined by $g(z) = \tilde{z}$.

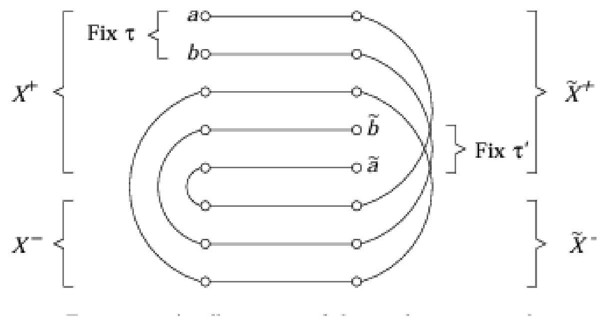

Figure 1. An illustration of the involution principle

4.1.3. *Sieve-Equivalence.* There is a variation of the involution principle that is concerned with "sieve-equivalence." We will mention only the simplest case here. Suppose that X and $\tilde{X}$ are (disjoint) finite sets. Let $Y \subseteq X$ and $\tilde{Y} \subseteq \tilde{X}$, and suppose that we are given bijections $f : X \to \tilde{X}$ and $g : Y \to \tilde{Y}$. Hence $\#(X - Y) = \#(\tilde{X} - \tilde{Y})$, and we wish to construct an explicit bijection h between $X - Y$ and $\tilde{X} - \tilde{Y}$. Pick $x \in X - Y$. As in the involution principle there will be a nonnegative integer n for which

$$f(g^{-1}f)^n(x) \in \tilde{X} - \tilde{Y}.$$

In this case n is unique since if $x \in \tilde{X} - \tilde{Y}$ then $g^{-1}(y)$ is undefined. Define $h(x)$ to be

$$f(g^{-1}f)^n(x)$$

where n satisfies (2.33). One easily checks that $h : X - Y \to \tilde{X} - \tilde{Y}$ is a bijection.

4.1.4. *Example 2.6.1.* Let Y be the set of all permutations $w \in S_n$ that fix 1, that is, $w(1) = 1$. Let $\tilde{Y}$ be the set of all permutations $w \in S_n$ with exactly one cycle. Thus $\#Y = \#\tilde{Y} = (n-1)!$, so

$$\#(S_n - Y) = \#(S_n - \tilde{Y}) = n! - (n-1)!.$$

It may not be readily apparent, however, how to construct a bijection h between $S_n - Y$ and $S_n - \tilde{Y}$. On the other hand, it is easy to construct a bijection g between Y and $\tilde{Y}$; namely, if $w = 1a_2 \cdots a_n \in Y$ (where w is written as a word, i.e., $w(i) = a_i$), then set

$$g(w) = (1, a_2, \dots, a_n)$$

(written as a cycle). Of course, we choose the bijection $f : S_n \to S_n$ to be the identity. Then equation (2.33) defines the bijection $h : S_n - Y \to S_n - \tilde{Y}$. For example, when $n = 3$ we depict f by solid lines and g by broken lines in Figure 2.3. Hence (writing permutations in the domain as words and in the range as products of cycles),

$$h(213) = (12)(3)h(231) = (1)(2)(3)h(312) = (1)(23)h(321) = (13)(2).$$

It is natural to ask here (and in other uses of the involution and related principles) whether there is a more direct description of h. In this example, there is little difficulty because Y and $\tilde{Y}$ are disjoint subsets (when $n \geq 2$) of the same set S_n. This special situation yields

$$h(w) = \begin{cases} w, & \text{if } w \notin \tilde{Y} \\ g^{-1}(w), & \text{if } w \in Y. \end{cases}$$

In this section, we have explored how involutions can be used to establish bijections, introduced the involution principle, and went through several examples to show its sieving process and forms. Next, we will look at the Garsia-Milne sieve.

4.2. **Garsia-Milne Sieve.** In essence, The Garsia-Milne Sieve, also known as the Garsia-Milne Involution Principle, takes two involutions on the same set satisfying certain requirements andf uses them to construct a bijection between the fixed points sets of two involutions. The involution principle was developed in 1980 by A. Garsia and S. Milne specifically to produce bijective proofs of the Rogers-Ramanujan Identities.

Specifically, the principle states:

Theorem 4.4. *Let $C = C^+ \cup C^-$ (where $C^+ \cap C^- = \emptyset$) be the disjoint union of two finite components C^+ and C^-. Let α and β be two involutions on C, each of whose fixed points lie in C^+. Let F_α (respectively, F_β) denote the fixed point set of α (respectively, β). Stipulate that $\alpha(C^+ - F_\alpha) \subset C^-$ and $\alpha(C^-) \subset C^+$, and similarly $\beta(C^+ - F_\beta) \subset C^-$ and $\beta(C^-) \subset C^+$. This means that outside the fixed point sets, both α and β map each component into the other.*

Then either a cycle of the permutation $\Delta = \alpha\beta$ contains no fixed points of either α or β, or it contains exactly one element of F_α and one of F_β.

The main idea behind the Garsia-Milne Involution is to construct a bijection between the fixed point sets F_α and F_β of the two involutions. While a proof of this is not given, the main idea is to map elements from the fixed point set of α to the fixerd point set of β, then to construct the inverse of the matching function by exchanging α and β.

Through this construction of bijections, the Garsia-Milne Involution has practical applications in combinatorial proofs, such as the previously mentioned Rogers-Ramanujan identities which discussed partition theory and q-series. Additionally, this principle can be extended to other combinatorial structures that require the construction of bijections between fixed point sets.

5. Further Directions & Acknowledgements

The exploration of sieve methods reveals numerous avenues for further study and application beyond combinatorics. Sieve methods have significant applications in graph theory and number theory. For instance, the classical Eratosthenes sieve is important for problems related to prime numbers, while more sophisticated sieves like the Turán sieve and the Brun sieve provide deeper insights into graph theory and prime distributions. These advanced sieve methods offer powerful tools for tackling longstanding problems and conjectures while also opening up new research directions.

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