# From Calculus to Analysis: The Stone-Weierstrass Theorem

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# Sequences/Series of Numbers

$$\{a_n\}$$

$$a_1, a_2, a_3, \ldots \to a$$

# Defining Sequence Convergence

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Can we do this for functions?

# Sequences of Functions

$$\{f_n\}$$

$$f_1, f_2, f_3, \ldots \to f$$

# Defining Pointwise Convergence

A sequence of functions $\{f_n\}$ defined on a set $E$ converges to a function $f$ on $E$ iff

$$\forall x \in E, \ \lim_{n \to \infty} f_n(x) = f(x)$$

# Defining Pointwise Convergence

A sequence of functions $\{f_n\}$ defined on a set $E$ converges to a function $f$ on $E$ iff

$$\forall x \in E,\ \varepsilon > 0\ \exists\ N \in \mathbb{Z}^+ : n > N \implies |f_n(x) - f(x)| < \varepsilon$$

# Issues with Pointwise Convergence

It would be useful for this convergence process to conserve certain properties, such as continuity or differentiability.

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However, this is not the case.

# Example 1

Let

$$f_n(x) = x^n$$

on $[0, \infty)$. Each $f_n$ is a polynomial, so it is continuous and differentiable. Now take

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n$$

on the same interval.

# Example 1

$$f(x) = \lim_{n \to \infty} x^n$$

If $0 \leq x < 1$, we have

$$f(x) = \lim_{n \to \infty} x^n = 0$$

If $x = 1$, we have

$$f(x) = \lim_{n \to \infty} x^n = 1$$

If $x > 1$, we have

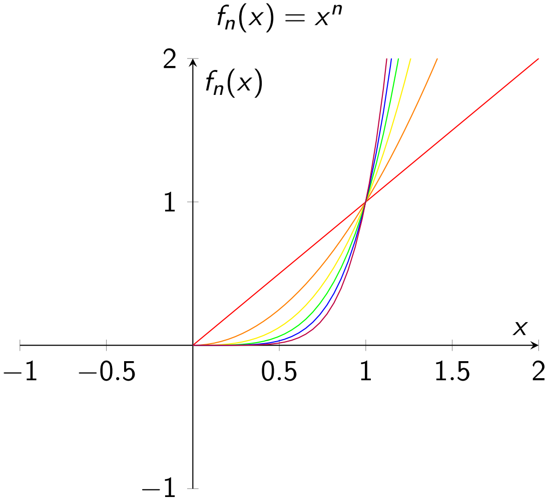
$$f(x) = \lim_{n \to \infty} x^n = \infty$$

Thus, we have

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \\ \infty, & x > 1 \end{cases}$$

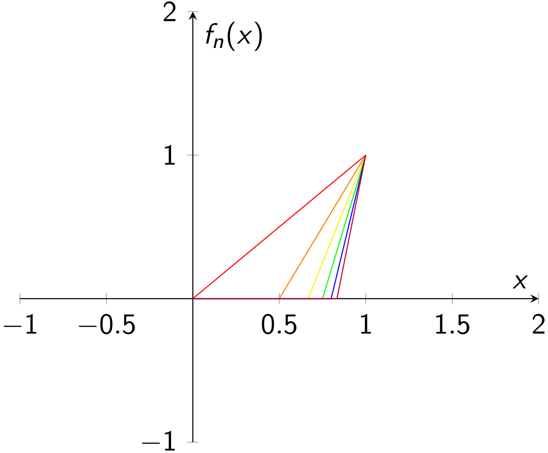
which is not continuous or differentiable

# Example 1



# Example 2

$$f_n(x) = \max((0, n(x-1)+1))$$



# Example 3

Let

$$f_n(x) = \arctan(nx)$$

on $(-\infty, \infty)$. $\arctan(x)$ is continuous and differentiable, so each $f_n$ is continuous and differentiable.
Now take

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \arctan(nx) = \arctan(\lim_{n\to\infty} nx)$$

on the same interval.

# Example 3

$$f(x) = \lim_{n\to\infty} \arctan(nx)$$

If $x < 0$,

$$f(x) = \lim_{n\to\infty} \arctan(nx) = \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

If $x = 0$,

$$f(x) = \lim_{n\to\infty} \arctan(nx) = \lim_{x\to\infty} \arctan(0) = 0$$

If $x > 0$,

$$f(x) = \lim_{n\to\infty} \arctan(nx) = \lim_{x\to\infty} \arctan(-x) = -\frac{\pi}{2}$$

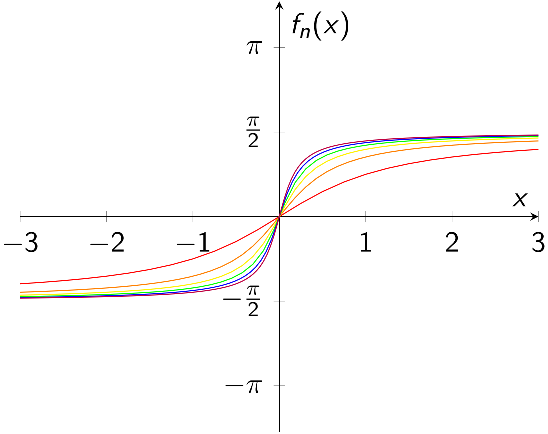
Thus,

$$f(x) = \begin{cases} -\frac{\pi}{2}, & x < 0 \\ 0, & x = 0 \\ \frac{\pi}{2}, & x > 0 \end{cases}$$

which is not continuous or differentiable

# Example 3

$$f_n(x) = \arctan(nx)$$



# Issues with Pointwise Convergence

If a sequence of functions $\{f_n\}$ is continuous/differentiable on a set $E$ and converges pointwise to a function $f$ on $E$, that does not guarantee that $f$ is continuous/differentiable.

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Is there another, stronger statement we can make about $\{f_n\}$ and $f$?

# Defining Uniform Convergence

A sequence of functions $\{f_n\}$ defined on a set $E$ converges **uniformly** to a function $f$ on $E$ iff

$$\forall \varepsilon > 0 \ \exists \ N \in \mathbb{Z}^+ : n > N, \ x \in E \implies |f_n(x) - f(x)| < \varepsilon$$

# Pointwise Convergence vs Uniform Convergence

A sequence of functions $\{f_n\}$ defined on a set $E$ converges to a function $f$ on $E$ iff

$$\forall x \in E,\ \varepsilon > 0\ \exists\ N \in \mathbb{Z}^+ : n > N \implies |f_n(x) - f(x)| < \varepsilon$$

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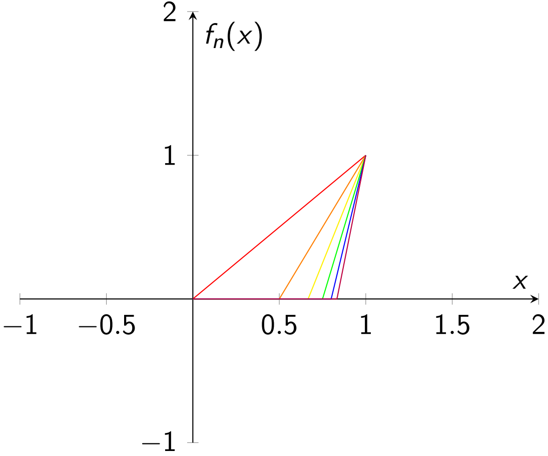
$$\forall \varepsilon > 0\ \exists\ N \in \mathbb{Z}^+ : n > N,\ x \in E \implies |f_n(x) - f(x)| < \varepsilon$$

# Pointwise Convergence vs Uniform Convergence

$$f_n(x) = \max((0, n(x-1)+1))$$



# Uniform Convergence Preserves Certain Properties

1. If a sequence of differentiable functions $\{f_n\}$ converges uniformly to $f$, and the sequence of derivatives $\{f_n'\}$ converges uniformly to $g$, then

$$f'(x) = \lim_{n\to\infty} f_n'(x) = g(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_n(x) = g(x)$$

2. If a sequence of integrable functions $\{f_n\}$ converges uniformly to $f$, then

$$\int_a^b f(x) = \lim_{n\to\infty} \int_a^b f_n(x)$$

$$\int_a^b \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_a^b f_n(x)$$

# The Stone-Weierstrass Theorem

If $f$ is a continuous function on $[a, b]$, there exists a sequence of polynomials $\{P_n\}$ that converges uniformly to $f$ on $[a, b]$.