Integer Linear Programming

Suhaan Mobhani
suhaan@mobhani.com

Euler Circle

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Linear Programming Fundamentals

Linear programming (LP) is a mathematical method used for optimizing a linear objective function, subject to linear inequality constraints. It involves:

- An objective function to be maximized or minimized.
- A set of linear constraints that represent limitations or requirements.
- Non-negative variables.

Problem Definition

An Linear Programming problem can be formulated as:

$$\begin{aligned} \text{maximize} \quad & \zeta = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\ & x_j \geq 0 \end{aligned}$$

Where c_i, a_{ij}, and b_i are constants, and x_i are nondiscrete variables.

Minimization Problems

We can adapt the maximization technique for minimization problems:

- To minimize ζ, maximize $-\zeta$.

This method is intuitive and straightforward.

Solving the LP

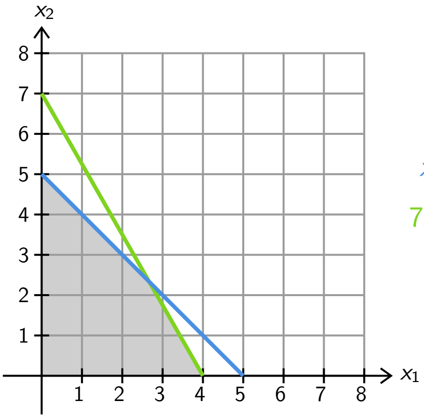
Consider the following example of a LP:

$$\begin{aligned} \text{maximize} \quad & \zeta = 4x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & 7x_1 + 4x_2 \leq 28, \\ & x_1, x_2 \geq 0. \end{aligned}$$

There are two inequalities which bound the problem.

Solving the LP

Maximize $\zeta = 4x_1 + 3x_2$.
A graphical representation:



$x_1 + x_2 \leq 5$

$7x_1 + 4x_2 \leq 28$

Feasability

Definition (Feasible Solution)

- Feasible solutions are the set of values for the decision variables that satisfy all the constraints.
- These solutions lie within the feasible region defined by the intersection of all the constraints.

Definition (Infeasible Solution)

- Any solution which is not considered feasible is an infeasible solution.
- These solutions lie outside the feasible region and do not represent valid solutions to the problem.

Solving the LP

By simple inspection, we can figure that the maximum lies at the intersection of the two inequalities.
So, we get that $\zeta = \frac{53}{3}$ when $x_1 = \frac{8}{3}$ and $x_2 = \frac{7}{3}$.
We define this particular solution that disregards the integer condition as the LP relaxation.

Solving the LP - Simplex Tableau

More sophisticated way than the graphical method that can be used for a larger number of variables.

$$\begin{aligned} \text{maximize} \quad & \zeta = 4x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & 7x_1 + 4x_2 \leq 28, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Convert it into standard form, we introduce slack variables s_1 and s_2:

$$\begin{aligned} x_1 + x_2 + s_1 &= 5, \\ 7x_1 + 4x_2 + s_2 &= 28, \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

Solving the LP - Simplex Tableau

Initial Simplex Tableau:

		x_1	x_2	s_1	s_2	
Basis	c_B	4	3	0	0	b
s_1	0	1	1	1	0	5
s_2	0	7	4	0	1	28
	z_j	0	0	0	0	0
	$c_j - z_j$	4	3	0	0	

Solving the LP - Simplex Tableau

Identify the entering variable x_1 (largest $c_j - z_j$) and the leaving variable (smallest non-negative ratio b_i/a_{ij}).

		$\mathbf{x_1}$	x_2	s_1	s_2		
Basis	c_B	**4**	3	0	0	b	Ratio
s_1	0	**1**	1	1	0	5	$\frac{5}{1} = 5$
s_2	0	**7**	4	0	1	28	$\frac{28}{7} = 4$
	z_j	**0**	0	0	0	0	
	$c_j - z_j$	**4**	3	0	0		

Definition (Pivot Element)

The Pivot Element is located at the intersection of the pivot column (of the entering variable) and the pivot row (of the leaving variable).

Solving the LP - Simplex Tableau

After swapping the variables in the basis, divide all elements in the pivot row by the pivot element.

		x_1	x_2	s_1	s_2	
Basis	c_B	4	3	0	0	b
s_1	0	1	1	1	0	5
x_1	4	7/7	4/7	0/7	1/7	28/7
	z_j	0	0	0	0	0
	$c_j - z_j$	4	3	0	0	

Notice how this alters the pivot element to have a unit value.

Solving the LP - Simplex Tableau

Now, we must find a value k such that in the pivot column:

$$s_1 - (k)(x_1) = 0.$$

Then, for every value in the s_1 row, we replace the original value s_1 with $s_1 - (k)(x_1)$ for the corresponding x_1 value.
Using $k = 1$ and carrying out the elementary row operations:

		x_1	x_2	s_1	s_2	
Basis	c_B	4	3	0	0	b
s_1	0	0	3/7	1	-1/7	1
x_1	4	1	4/7	0	1/7	4
	z_j	0	0	0	0	0
	$c_j - z_j$	4	3	0	0	

Solving the LP - Simplex Tableau

Now, populate the z_j row using the following for each column one at a time.

$$z_j = \overrightarrow{c_B} \cdot \begin{pmatrix} s_1 \\ x_1 \end{pmatrix}$$

Then, recalculate the $c_j - z_j$ row using the new z_j values

		x_1	x_2	s_1	s_2	
Basis	c_B	4	3	0	0	b
s_1	0	0	3/7	1	-1/7	1
x_1	4	1	4/7	0	1/7	4
	z_j	4	16/7	0	4/7	16
	$c_j - z_j$	0	5/7	0	-4/7	

Solving the LP - Simplex Tableau

We continue this process of elementary row operations. In this case, the leaving variable is s_1 and the entering variable is x_2. The pivot element is $3/7$.

Basis	c_B	x_1 4	x_2 3	s_1 0	s_2 0	b	Ratio
s_1	0	0	3/7	1	-1/7	1	7/3
x_1	4	1	4/7	0	1/7	4	7
	z_j	4	16/7	0	4/7	16	
	$c_j - z_j$	0	5/7	0	-4/7		

Solving the LP - Simplex Tableau

We divide all values in the x_2 row by the pivot element. Then, we use $k = 4/7$ and modify the x_1 row using $x_1^* = x_1 - (4/7)x_2$.

		x_1	x_2	s_1	s_2	
Basis	c_B	4	3	0	0	b
x_2	3	0	1	7/3	-1/3	7/3
x_1	4	1	0	-4/3	1/3	8/3
	z_j	4	16/7	0	4/7	16
	$c_j - z_j$	0	5/7	0	-4/7	

Solving the LP - Simplex Tableau

Now, we recalculate our z_j and $c_j - z_j$ values.

		x_1	x_2	s_1	s_2	
Basis	c_B	4	3	0	0	b
x_2	3	0	1	7/3	-1/3	7/3
x_1	4	1	0	-4/3	1/3	8/3
	z_j	4	3	5/3	1/3	53/3
	$c_j - z_j$	0	0	-5/3	-1/3	

Since all our $c_j - z_j$ values are nonpositive, we terminate the process. We have found our maximum $\zeta = 53/3$ when $x_1 = 8/3$ and $x_2 = 7/3$. This is our LP relaxation.

Defining an ILP

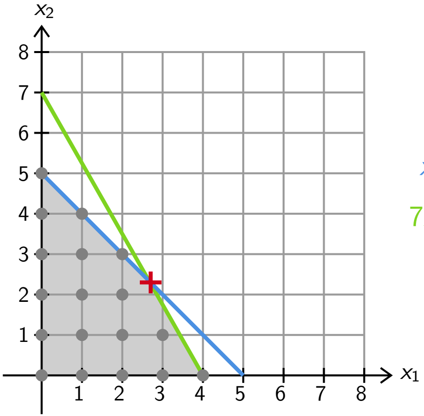
An Integer Linear Programming problem can be formulated as:

$$\begin{aligned} \text{maximize} \quad & \zeta = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\ & x_j \in \mathbb{Z}, \\ & x_j \geq 0 \quad \forall j. \end{aligned}$$

Where c_i, a_{ij}, and b_i are constants, and x_i are integer variables.

Extending to ILP

Integer Problem $\leq$ LP Relaxation



$x_1 + x_2 \leq 5$

$7x_1 + 4x_2 \leq 28$

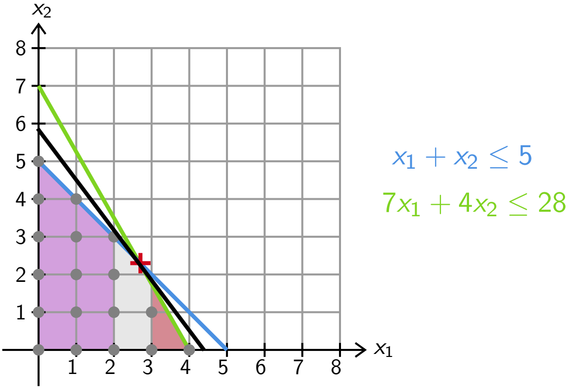
Branch and Bound method

1. **Branching:**
   - Select a variable with a noninteger value in the solution of the LP relaxation.
   - Create two subproblems or branches by adding constraints to the selected variable: $x \leq \lfloor x^* \rfloor$ and $x \geq \lceil x^* \rceil$.
2. **Bounding:**
   - Solve the LP relaxation of the new subproblems.
   - If a subproblem is infeasible, discard it.
   - If a subproblem has an integer solution, compare it with the best solution and update if it is better.
3. **Pruning:**
   - Discard subproblems that cannot produce a better solution than the best solution.
4. **Iteration:**
   - Repeat branching, bounding, and pruning steps until all subproblems are either solved or discarded.

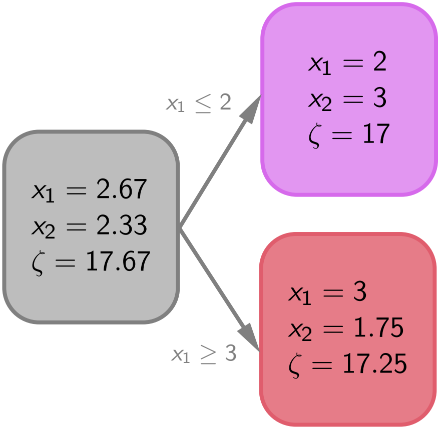
Branch and Bound method

$\zeta = \frac{53}{3}$ when $x_1 = \frac{8}{3}$ and $x_2 = \frac{7}{3}$
Branch on x_1 to get the following branches:

$$x_1 \leq 2 \qquad x_1 \geq 3.$$



Branch and Bound method



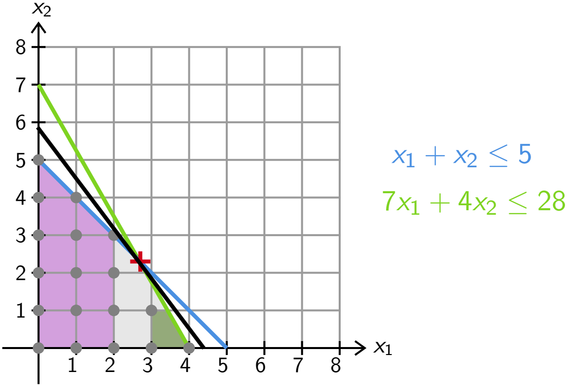
Since there is an integer solution,
BEST $= 17$

Since $\zeta = 17.25 >$ BEST,
Branch further on x_2
for the red branch

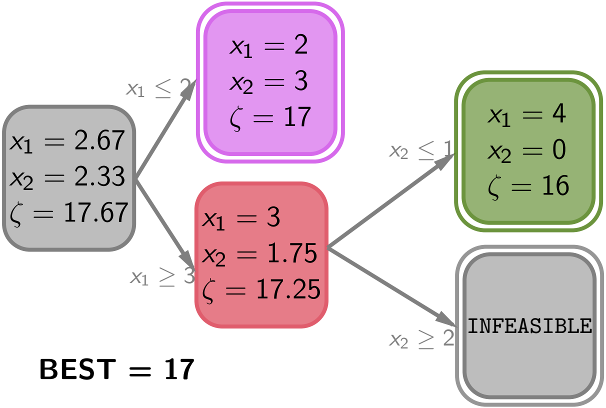
Branch and Bound method

Branch on x_2 to get the following branches:

$$x_2 \leq 1 \qquad x_2 \geq 2.$$



Branch and Bound method



Since $\zeta = 16 <$BEST, no further branched solution can be better

Cannot branch further since integer result

BEST = 17

Extensions on ILP

Mixed Integer Linear Programming (MILP) involves problems where only some of the variables are constrained to be integers, while others can be non-integers.

Binary (0-1) variables are decision variables that can only take on the values 0 or 1. These determine whether to include a certain non-decision variable in a solution (1) or not (0).

Integer Nonlinear Programming (INLP) involves optimization problems with nonlinear objective functions or constraints. (ie: degrees more than 1)

Applications of ILP

ILP robust applications in fields such as operations research, economics, and computer science due to its ability to efficiently optimize tasks. These applications are explained more in detail in my paper.

Thank You!

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