Stability Analysis in Dynamical Systems

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Dynamical Systems

Dynamical system: a system that evolves over time.

Described by:

$$\dot{x} = f(x(t), t)$$

where $x \in \mathbb{R}^n$ and $\dot{x} = \frac{dx}{dt}$.

Definition (Fixed Point)

A **fixed point** (or equilibrium point), x^*, is a state where $f(x^*, t) = 0$.

Stability

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A fixed point x^* is attracting if there exists $\delta > 0$ such that if $\|x(0) - x^*\| < \delta$ then

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Definition (Asymptotically Stable)

A fixed point x^* is asymptotically stable if it is stable and attracting.

Linear Systems

Linear System:

$$\dot{x} = Ax$$

where $A \in \mathbb{R}^{n \times n}$.

Theorem

A fixed point x^* of the linear system $\dot{x} = Ax$ is asymptotically stable if all eigenvalues of A have negative real parts.

Linearization

Given

$$\dot{x} = f(x)$$

define perturbation

$$u = x - x^*$$

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Taylor expansion:

$$\dot{u} = f(x^*) + J(x^*)u + \text{higher order terms}$$

$$\text{higher order terms} \approx 0$$

$$\dot{u} \approx f(x^*) + J(x^*)u = J(x^*)u$$

$$\dot{x} \approx J(x^*)(x - x^*)$$

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If the eigenvalues of $J(x^*)$ are negative then the fixed point $x = x^*$ is asymptotically stable.

Control Theory

Dynamics of a controlled system:

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where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

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$$\dot{x} = Ax + Bu$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

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Feedback control:

$$u = -Kx$$

where $K \in \mathbb{R}^{m \times n}$. So

$$\dot{x} = Ax - BKx = (A - BK)x.$$

Choose K such that the eigenvalues of $A - BK$ are negative.

Spring-Mass-Damper System

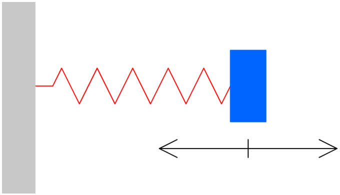

Figure: Spring-Mass-Damper

Spring-mass-damper system:

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Spring-Mass-Damper System (cont.)

Let

$$u = -Kx$$

where $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$.
Evaluating $A - BK$:

$$A - BK = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 - k_1 & 3 - k_2 \end{pmatrix}.$$

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Let eigenvalues be λ. We have that

$$\lambda^2 + (k_2 - 3)\lambda + (k_1 + 2) = 0.$$

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Arbitrarily choose $\lambda = -2, -5$. They satisfy

$$(\lambda + 2)(\lambda + 5) = \lambda^2 + 7\lambda + 10 = 0.$$

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$$k_1 + 2 = 10 \implies k_1 = 8,$$
$$k_2 - 3 = 7 \implies k_2 = 10.$$

So $K = \begin{pmatrix} 8 & 10 \end{pmatrix}$.

Nonlinear Dynamics and Control

Given

$$\dot{x} = f(x, u), \quad f(x^*, u^*) = 0,$$

define perturbations

$$\Delta x = x - x^*, \quad \Delta u = u - u^*.$$

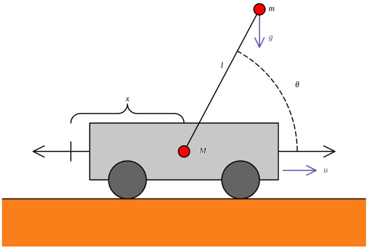
Approximate using Taylor expansion:

$$\Delta\dot{x} = \dot{x} = f(x, u) \approx f(x^*, u^*) + A\Delta x + B\Delta u,$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x^*, u^*)}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x^*, u^*)}$$

$$\Delta\dot{x} \approx A\Delta x + B\Delta u$$

Cart-Pole Problem



Dynamics of the cart-pole problem:

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = u$$
$$l\ddot{\theta} + g\sin(\theta) = \ddot{x}\cos(\theta).$$

The variables are:

- x: Position of the cart
- θ: Angle of the pole (upright is $\theta = 0$)
- u : Control force applied to the cart
- M : Mass of the cart
- m : Mass of the pole
- l : Length of the pole
- g : Acceleration due to gravity

Cart-Pole Problem (cont.)

State vector:

$$\mathbf{x} = (x, \dot{x}, \theta, \dot{\theta})^T \approx (0, 0, 0, 0)^T.$$

We can make the approximations

$$\sin(\theta) \approx \theta,$$
$$\cos(\theta) \approx 1,$$
$$\dot{\theta}^2 \approx 0.$$

Substituting gives us

$$(M + m)\ddot{x} + ml\ddot{\theta} \approx u$$
$$l\ddot{\theta} + g\theta \approx \ddot{x}.$$

Cart-Pole Problem (cont.)

$$(M+m)\ddot{x} + ml\ddot{\theta} \approx u$$
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Linearization gives us

$$\dot{\mathbf{x}} \approx A\mathbf{x} + Bu$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M+2m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{Mg+mg}{l(M+2m)} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{1}{M+2m} \\ 0 \\ \frac{1}{l(M+2m)} \end{pmatrix}.$$

Let

$$u = -Kx$$

and find eigenvalues of $A - BK$ (requires numerical methods).

Thank You

Thank you for listening!