Generalizing Fourier Series to Groups

Siddharth Kothari
spacersid1331@gmail.com

Euler Circle

12th July 2024

Prerequisites

1. Linear Algebra
2. Group Theory
   - Definition of a **group**
   - Examples of groups, including the **cyclic group** $\mathbb{Z}_n$
   - Subgroups
   - Group **homomorphisms** and **isomorphisms**
   - **Group actions** and **conjugacy classes**.
   - **Direct sum** of groups.

An Elementary Geometry Theorem

Theorem

Given a point and a line, the shortest distance between them is the length of the segment connecting the two such that it's perpendicular to the given line.

An Elementary Geometry Theorem

Theorem

Given a point and a line, the shortest distance between them is the length of the segment connecting the two such that it's perpendicular to the given line.

Remark

Arguably, we can't do much with this theorem, **at least in this setting**.

Hilbert Spaces

Hilbert Spaces

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.

Hilbert Spaces

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.
It's is an ordinary vector space, V, plus

Hilbert Spaces

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.
It's is an ordinary vector space, V, plus

- An **inner product** on V, commonly denoted by $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$. It's the generalization of the dot product on $\mathbb{R}^n$. Just like how the **dot product** measures the angle or closeness between two vectors in $\mathbb{R}^n$, the general inner product does the same for vectors in an arbitrary vector space V.

Hilbert Spaces

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.
It's is an ordinary vector space, V, plus

- An **inner product** on V, commonly denoted by $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$. It's the generalization of the dot product on $\mathbb{R}^n$. Just like how the **dot product** measures the angle or closeness between two vectors in $\mathbb{R}^n$, the general inner product does the same for vectors in an arbitrary vector space V.
  1. **Positive definiteness:** For all $v \in V$, we have $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \iff v = 0$.
  2. **Linearity in the first argument:** For all $v, u, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have $\langle \alpha v + \beta u, w \rangle = \alpha \langle v, w \rangle + \beta \langle u, w \rangle$.
  3. **Conjugate symmetry:** For all $v, w \in V$, we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

Hilbert Spaces

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.
It's is an ordinary vector space, V, plus

- An **inner product** on V, commonly denoted by $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$. It's the generalization of the dot product on $\mathbb{R}^n$. Just like how the **dot product** measures the angle or closeness between two vectors in $\mathbb{R}^n$, the general inner product does the same for vectors in an arbitrary vector space V.
    1. **Positive definiteness:** For all $v \in V$, we have $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \iff v = 0$.
    2. **Linearity in the first argument:** For all $v, u, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have $\langle \alpha v + \beta u, w \rangle = \alpha \langle v, w \rangle + \beta \langle u, w \rangle$.
    3. **Conjugate symmetry:** For all $v, w \in V$, we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
- **Technical condition:** The inner product induces a **norm** on V, given by $\|v\| = \sqrt{\langle v, v \rangle}$ such that V is **complete** with respect to the **metric** induced by the norm, $d(x, y) = \|x - y\|$ for all $x, y \in \mathcal{H}$.

Hilbert Spaces

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.
It's is an ordinary vector space, V, plus

- An **inner product** on V, commonly denoted by $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$. It's the generalization of the dot product on $\mathbb{R}^n$. Just like how the **dot product** measures the angle or closeness between two vectors in $\mathbb{R}^n$, the general inner product does the same for vectors in an arbitrary vector space V.
  1. **Positive definiteness:** For all $v \in V$, we have $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \iff v = 0$.
  2. **Linearity in the first argument:** For all $v, u, w \in V$ and $\alpha, \beta \in \mathbb{F}$, we have $\langle \alpha v + \beta u, w \rangle = \alpha \langle v, w \rangle + \beta \langle u, w \rangle$.
  3. **Conjugate symmetry:** For all $v, w \in V$, we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
- **Technical condition:** The inner product induces a **norm** on V, given by $\|v\| = \sqrt{\langle v, v \rangle}$ such that V is **complete** with respect to the **metric** induced by the norm, $d(x, y) = \|x - y\|$ for all $x, y \in \mathcal{H}$.

For our purposes here,

$$\text{A Hilbert Space} = \text{An Inner Product Space.}$$

Hilbert Spaces: Orthogonality

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.

Definition

Given a Hilbert space $\mathcal{H}$, two vectors $x, y \in \mathcal{H}$ are said to be orthogonal if $\langle x, y \rangle = 0$.

Hilbert Spaces: Orthogonality

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.

Definition

Given a Hilbert space $\mathcal{H}$, two vectors $x, y \in \mathcal{H}$ are said to be orthogonal if $\langle x, y \rangle = 0$.

Remark

Note how a property was turned into a definition!

Hilbert Spaces: Orthogonality

A **Hilbert space** is the general setting to talk about **lengths** and **angles**.

Definition

Given a Hilbert space $\mathcal{H}$, two vectors $x, y \in \mathcal{H}$ are said to be orthogonal if $\langle x, y \rangle = 0$.

Remark

Note how a property was turned into a definition!

Definition

Given a Hilbert space $\mathcal{H}$ and a subspace S, a vector $v \in \mathcal{H}$ is said to be orthogonal to P if $\langle v, x \rangle = 0$ for all $x \in S$.

Hilbert Spaces: First Examples

Example

The real euclidean space, $\mathbb{R}^n$, with the dot product as the inner product.

Hilbert Spaces: First Examples

Example

The real euclidean space, $\mathbb{R}^n$, with the dot product as the inner product.

Example

The set of **continuous** functions $f : [a, b] \to \mathbb{R}$ with

- Addition and scalar multiplication given point wise.
- The inner product is given by:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx \text{ for all } f, g \in \mathcal{H}.$$

The Best Approximation Theorem

The Best Approximation Theorem: A Question

Question

Given a (closed) subspace P *of a Hilbert space* $\mathcal{H}$ *and a vector* $v \in \mathcal{H}$*, find the vector* $p \in P$ *such that* $\|p - v\| = \min_{x \in P} \|x - v\|$.

The Best Approximation Theorem: A Question

Question

Given a (closed) subspace P of a Hilbert space $\mathcal{H}$ and a vector $v \in \mathcal{H}$, find the vector $p \in P$ such that $\|p - v\| = \min_{x \in P} \|x - v\|$.

Remark

The vector p defined above is called the projection of v onto the subspace P, denoted by $\text{proj}_P(v)$.

The Best Approximation Theorem: A Question

Question

Given a (closed) subspace P of a Hilbert space $\mathcal{H}$ and a vector $v \in \mathcal{H}$, find the vector $p \in P$ such that $\|p - v\| = \min_{x \in P} \|x - v\|$.

Remark

The vector p defined above is called the projection of v onto the subspace P, denoted by $\mathrm{proj}_P(v)$.

Proposition

The vector $p \in P$ exists and is such that $\langle v - p, x \rangle = 0$ for all $x \in P$. In other words, the error vector $e = v - p$ is orthogonal to the subspace P.

Note how close this feels to the elementary theorem!

The Best Approximation Theorem: A Question

Question

Given a (closed) subspace P of a Hilbert space $\mathcal{H}$ and a vector $v \in \mathcal{H}$, find the vector $p \in P$ such that $\|p - v\| = \min_{x \in P} \|x - v\|$.

Remark

The vector p defined above is called the projection of v onto the subspace P, denoted by $\mathrm{proj}_P(v)$.

Proposition

The vector $p \in P$ exists and is such that $\langle v - p, x \rangle = 0$ for all $x \in P$. In other words, the error vector $e = v - p$ is orthogonal to the subspace P.

Note how close this feels to the elementary theorem!

Remark

Let $\{e_i\}_{i=1}^n$ is a **orthogonal basis** for P. Then, $\mathrm{proj}_P(v) = \sum_{i=1}^n \frac{\langle e_i, v \rangle}{\langle e_i, e_i \rangle} e_i$.

The Best Approximation Theorem: A Proof

Proof.

First, we show that p such that $v - p \perp P$ is indeed the vector that minimizes the distance between P and v, that is $\|p - v\| = \min_{x \in P} \|x - v\|$.

The Best Approximation Theorem: A Proof

Proof.

First, we show that p such that $v - p \perp P$ is indeed the vector that minimizes the distance between P and v, that is $\|p - v\| = \min_{x \in P} \|x - v\|$. Start by noting that $x \in P$ means there exists a $y \in P$ such that $x = p + y$.

The Best Approximation Theorem: A Proof

Proof.

First, we show that p such that $v - p \perp P$ is indeed the vector that minimizes the distance between P and v, that is $\|p - v\| = \min_{x \in P} \|x - v\|$. Start by noting that $x \in P$ means there exists a $y \in P$ such that $x = p + y$. Expanding $\|x - v\|^2$,

$$\|x - v\|^2 = \|(p + y) - v\|^2 = \|(p - v) + y\|^2.$$

The Best Approximation Theorem: A Proof

Proof.

First, we show that p such that $v - p \perp P$ is indeed the vector that minimizes the distance between P and v, that is $\|p - v\| = \min_{x \in P} \|x - v\|$. Start by noting that $x \in P$ means there exists a $y \in P$ such that $x = p + y$. Expanding $\|x - v\|^2$,

$$\|x - v\|^2 = \|(p + y) - v\|^2 = \|(p - v) + y\|^2.$$

Next , $p - v \perp y \implies \|(p - v) + y\|^2 = \|p - v\|^2 + \|y\|^2$.

The Best Approximation Theorem: A Proof

Proof.

First, we show that p such that $v - p \perp P$ is indeed the vector that minimizes the distance between P and v, that is $\|p - v\| = \min_{x \in P} \|x - v\|$. Start by noting that $x \in P$ means there exists a $y \in P$ such that $x = p + y$. Expanding $\|x - v\|^2$,

$$\|x - v\|^2 = \|(p + y) - v\|^2 = \|(p - v) + y\|^2.$$

Next , $p - v \perp y \implies \|(p - v) + y\|^2 = \|p - v\|^2 + \|y\|^2$. Thus

$$\|x - v\|^2 = \|p - v\|^2 + \|y\|^2 \implies \|x - v\|^2 \geq \|p - v\|^2.$$

The Best Approximation Theorem: A Proof

Proof.

First, we show that p such that $v - p \perp P$ is indeed the vector that minimizes the distance between P and v, that is $\|p - v\| = \min_{x \in P} \|x - v\|$. Start by noting that $x \in P$ means there exists a $y \in P$ such that $x = p + y$. Expanding $\|x - v\|^2$,

$$\|x - v\|^2 = \|(p + y) - v\|^2 = \|(p - v) + y\|^2.$$

Next , $p - v \perp y \implies \|(p - v) + y\|^2 = \|p - v\|^2 + \|y\|^2$. Thus

$$\|x - v\|^2 = \|p - v\|^2 + \|y\|^2 \implies \|x - v\|^2 \geq \|p - v\|^2.$$

Clearly, we have equality if and only if $y = 0 \iff x = p$. ∎

The Best Approximation Theorem: A Proof

Proof.

Then, it's just a matter of algebraic manipulation to show $e = v - \sum_{i=1}^{n} \frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} e_i$ is indeed orthogonal to P. Start by expanding the expression for $\langle v - p, x\rangle$,

$$\left\langle v - \sum_{i=1}^{n} \frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} e_i, x \right\rangle = \left\langle v - \sum_{i=1}^{n} \frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} e_i, \sum_{j=1}^{n} a_j e_j \right\rangle = \sum_{j=1}^{n} \overline{a_j} \left\langle v - \sum_{i=1}^{n} \frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} e_i, e_j \right\rangle.$$

Now, note that

$$\begin{aligned}\left\langle v - \sum_{i=1}^{n} \frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} e_i, e_j \right\rangle &= \langle v, e_j\rangle - \left\langle \sum_{i=1}^{n} \frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} e_i, e_j \right\rangle \\ &= \langle v, e_j\rangle - \sum_{i=1}^{n} \frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} \langle e_i, e_j\rangle = \langle v, e_j\rangle - \langle v, e_j\rangle \\ &= 0.\end{aligned}$$

Hence $\langle v - p, x\rangle = 0$ for all $x \in P$. ■

The Best Approximation Theorem: An Example

- Let's take $\mathcal{H} = L([-\pi, \pi])$ to be the Hilbert space of continuous real valued functions defined on $[-\pi, \pi]$, with the inner product we defined earlier.

The Best Approximation Theorem: An Example

- Let's take $\mathcal{H} = L([-\pi, \pi])$ to be the Hilbert space of continuous real valued functions defined on $[-\pi, \pi]$, with the inner product we defined earlier.
- Next take P to be the infinite dimensional subspace generated by orthonormal set

$$C = \left\{ \frac{1}{\sqrt{\pi}} \cos(nx) : n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{2\pi}} \right\} \subset \mathcal{H}.$$

The Best Approximation Theorem: An Example

- Let's take $\mathcal{H} = L([-\pi, \pi])$ to be the Hilbert space of continuous real valued functions defined on $[-\pi, \pi]$, with the inner product we defined earlier.
- Next take P to be the infinite dimensional subspace generated by orthonormal set

$$C = \left\{ \frac{1}{\sqrt{\pi}} \cos(nx) : n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{2\pi}} \right\} \subset \mathcal{H}.$$

- For simplicity, let's start with $v = x^2$,

The Best Approximation Theorem: An Example

- Let's take $\mathcal{H} = L([-\pi, \pi])$ to be the Hilbert space of continuous real valued functions defined on $[-\pi, \pi]$, with the inner product we defined earlier.
- Next take P to be the infinite dimensional subspace generated by orthonormal set

$$C = \left\{ \frac{1}{\sqrt{\pi}} \cos(nx) : n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{2\pi}} \right\} \subset \mathcal{H}.$$

- For simplicity, let's start with $v = x^2$,

$$\begin{aligned} \mathrm{proj}_{\mathrm{span}(C)}(x^2) &= \left\langle x^2, \frac{1}{\sqrt{2\pi}} \right\rangle + \sum_{i=1}^{\infty} \langle x^2, \frac{1}{\sqrt{\pi}} \cos(ix) \rangle \frac{1}{\sqrt{\pi}} \cos(ix) \\ &= \frac{1}{\pi} \sum_{i=0}^{\infty} \int_{-\pi}^{\pi} x^2 \cos(ix)\, dx \cos(ix) \\ &= \frac{\pi^2}{3} + \sum_{i=1}^{\infty} \frac{4}{i^2} (-1)^i \cos(ix). \end{aligned}$$

Which you might recognize as the **Fourier series** of x^2 from $-\pi$ to π!

The Failure of the Cosine Basis

Example

Next we use C to approximate $f(x) = x$, meaning we compute

$$\mathrm{proj}_{\mathrm{span}(C)}(x) = \left\langle x, \frac{1}{\sqrt{2\pi}} \right\rangle + \frac{1}{2\pi} \sum_{n=1}^{\infty} \langle x, \cos(nx) \rangle \cos(nx).$$

- Before diving right in to bash out the integral $\int_{-\pi}^{\pi} x\cos(nx)\,dx$, note that $x\cos(nx)$ is an odd function and the domain of integration is symmetric about the origin. Hence the positive and negative areas on either side of the $y-$axis will completely cancel, meaning $\langle x, \cos(nx) \rangle = 0$ for all $n \in \mathbb{N}$.
- Reasoning identically, $\langle x, \frac{1}{\sqrt{2\pi}} \rangle = \int_{-\pi}^{+\pi} \frac{x}{\sqrt{2\pi}}\,dx = 0$ which means $\mathrm{proj}_{\mathrm{span}(C)}(x) = 0$! C has failed!

The Best Approximation Theorem & Fourier Series

As you might have guessed, we'll need the sines as well to approximate any function.

The Best Approximation Theorem & Fourier Series

As you might have guessed, we'll need the sines as well to approximate any function. Once we do rope them in, we're left with the basis

$$T = S \cup C = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx) : n \in \mathbb{N} \right\}.$$

The Best Approximation Theorem & Fourier Series

As you might have guessed, we'll need the sines as well to approximate any function. Once we do rope them in, we're left with the basis

$$T = S \cup C = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx) : n \in \mathbb{N} \right\}.$$

Projecting functions onto the span of this new basis gives us

$$\boxed{A_0 + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]}$$

The Best Approximation Theorem & Fourier Series

As you might have guessed, we'll need the sines as well to approximate any function. Once we do rope them in, we're left with the basis

$$T = S \cup C = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx) : n \in \mathbb{N} \right\}.$$

Projecting functions onto the span of this new basis gives us

$$\boxed{A_0 + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]}$$

where

- $A_n = \frac{1}{\sqrt{\pi}} \cos(nx) \langle f, \frac{1}{\sqrt{\pi}} \cos nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$ for all $n \in \mathbb{N}$.
- $B_n = \frac{1}{\sqrt{\pi}} \sin(nx) \langle f, \frac{1}{\sqrt{\pi}} \sin nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$ for all $n \in \mathbb{N}$.
- $A_0 = \frac{1}{\sqrt{2\pi}} \langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$

The Best Approximation Theorem & Fourier Series

As you might have guessed, we'll need the sines as well to approximate any function. Once we do rope them in, we're left with the basis

$$T = S \cup C = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx) : n \in \mathbb{N} \right\}.$$

Projecting functions onto the span of this new basis gives us

$$\boxed{A_0 + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx]}$$

where

- $A_n = \frac{1}{\sqrt{\pi}} \cos(nx) \langle f, \frac{1}{\sqrt{\pi}} \cos nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$ for all $n \in \mathbb{N}$.
- $B_n = \frac{1}{\sqrt{\pi}} \sin(nx) \langle f, \frac{1}{\sqrt{\pi}} \sin nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$ for all $n \in \mathbb{N}$.
- $A_0 = \frac{1}{\sqrt{2\pi}} \langle f, \frac{1}{\sqrt{2\pi}} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$,

which is the Fourier series of a function f defined on the compact interval $[-\pi, +\pi]$.

The Trigonometric Basis

Remark

As we saw, the Fourier series of a function boils down to computing it's projection onto the span of T, which consists of all linear combinations of the trigonometric polynomials.

The Trigonometric Basis

Remark

As we saw, the Fourier series of a function boils down to computing it's projection onto the span of T, which consists of all linear combinations of the trigonometric polynomials.

Question

How can we be sure that we can approximate any function f using T?

How Can We Take This Further?

How Can We Take This Further?

Question

How would one go about computing the Fourier series of a function $f : G \to \mathbb{C}$*, where* G *is an arbitrary group? Is it even possible to construct such an analogue?*

The Group Algebra $\mathbb{C}[G]$

The setting here would be the vector space of functions $f : G \to \mathbb{C}$, denoted by $\mathbb{C}[G]$.

- **Operations:** Addition and scalar multiplication are given pointwise as usual.

The Group Algebra $\mathbb{C}[G]$

The setting here would be the vector space of functions $f : G \to \mathbb{C}$, denoted by $\mathbb{C}[G]$.

- **Operations:** Addition and scalar multiplication are given pointwise as usual.
- **Inner product:** We equip it with the most 'natural' inner product defined by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)} \text{ for all } f_1, f_2 \in \mathbb{C}[G].$$

The Group Algebra $\mathbb{C}[G]$

The setting here would be the vector space of functions $f : G \to \mathbb{C}$, denoted by $\mathbb{C}[G]$.

- **Operations:** Addition and scalar multiplication are given pointwise as usual.
- **Inner product:** We equip it with the most 'natural' inner product defined by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)} \text{ for all } f_1, f_2 \in \mathbb{C}[G].$$

Recall the inner products on $\mathbb{C}^n$ and $L([a, b])$,

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i \overline{w_i}, \qquad \langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

The Group Algebra $\mathbb{C}[G]$

The setting here would be the vector space of functions $f : G \to \mathbb{C}$, denoted by $\mathbb{C}[G]$.

- **Operations:** Addition and scalar multiplication are given pointwise as usual.
- **Inner product:** We equip it with the most 'natural' inner product defined by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)} \text{ for all } f_1, f_2 \in \mathbb{C}[G].$$

Recall the inner products on $\mathbb{C}^n$ and $L([a, b])$,

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i \overline{w_i}, \qquad \langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

Warning: the inner product on $\mathbb{C}[G]$ completely breaks down when $|G| = \infty$!

An Algebraic Interlude: Representation Theory

What is Representation Theory?

Roughly speaking, **representation theory** lies at the intersection of the two cornerstones of modern algebra: group theory, the study of symmetry and linear algebra, the study of maps that deform vector spaces in a certain, **clean way**.

What is Representation Theory?

Definition

A *representation* of a group G is an ordered pair (ρ, V), where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a homomorphism of groups.

Remark

- Note that $\mathrm{GL}(V)$ is the **general linear group** of V, the set of invertible $\dim V$ by $\dim V$ matrices which forms a group under the usual matrix multiplication.
- Remember from linear algebra that these matrices describe invertible linear maps from V to itself, so here we are viewing a $g \in G$ as *acting on a vector space* via an invertible linear map, namely, $\rho(g)$.

What is Representation Theory?

Definition

A *representation* of a group G is an ordered pair (ρ, V), where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a homomorphism of groups.

Example

Consider $G = \mathbb{Z}_n$, the cyclic group of order n. We could interpret the generator as a rotation by $2\pi/n$ about the origin, meaning G consists of the rotational symmetries of an $n-$gon, so it's reasonable to let $V = \mathbb{R}^2$, the plane, and

$$\rho(k) = \begin{bmatrix} \cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\ \sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{bmatrix}$$

for all $k \in \mathbb{Z}_n$, which is the matrix corresponding to the rotation.

Sub-representations

Just like in linear algebra we have subspaces and in group theory subgroups, we also have such a sub-structure for representations, called a subrepresentation.

Sub-representations

Just like in linear algebra we have subspaces and in group theory subgroups, we also have such a sub-structure for representations, called a subrepresentation.

For the purposes of this talk, a subrepresentation of a representation (ρ, V) is both a

- representation in it's own right and,
- sits inside the given representation (ρ, V).

Irreducible Representations

With subrepresentations, the concepts of reducible and irreducible representations naturally come up.

Definition

A representation (ρ, V) of a group G is said to be *irreducible* if its only subrepresentations are the trivial representation $W = \{0\}$, and itself. A representation is said to be *reducible* if it is not irreducible.

The Character of a Representation

Definition

Let (ρ, V) be a representation of a group G. The *character* of the representation (ρ, V) is a function $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \mathrm{Tr}\big(\rho(g)\big)$ for all $g \in G$. A character of an irreducible representation is said to be an irreducible character.

The Great Theorem

The Great Theorem on Orthogonality

Now we come to the central result of my presentation!

Theorem

Let G be a finite group and $\mathcal{C}_G$ the vector space of complex-valued class functions defined on G. Then the set of irreducible characters of G forms an orthonormal basis for $\mathcal{C}_G$.

Remark

A class function $f : G \to \mathbb{C}$ is a function that is constant on a conjugacy class. For finite ableian groups, like $\mathbb{Z}_n$, the notion of a class function and an ordinary function in $\mathbb{C}[G]$ are the same.

The Great Theorem on Orthogonality

Now we come to the central result of my presentation!

Theorem

Let G be a finite group and $\mathcal{C}_G$ the vector space of complex-valued class functions defined on G. Then the set of irreducible characters of G forms an orthonormal basis for $\mathcal{C}_G$.

Remark

A class function $f : G \to \mathbb{C}$ is a function that is constant on a conjugacy class. For finite ableian groups, like $\mathbb{Z}_n$, the notion of a class function and an ordinary function in $\mathbb{C}[G]$ are the same.

Due to time constraints, we won't prove this here. More importantly, we discuss why this theorem is a significant leap.

The Great Theorem on Orthogonality

Now we come to the central result of my presentation!

Theorem

Let G be a finite group and $\mathcal{C}_G$ the vector space of complex-valued class functions defined on G. Then the set of irreducible characters of G forms an orthonormal basis for $\mathcal{C}_G$.

Remark

A class function $f : G \to \mathbb{C}$ is a function that is constant on a conjugacy class. For finite ableian groups, like $\mathbb{Z}_n$, the notion of a class function and an ordinary function in $\mathbb{C}[G]$ are the same.

Due to time constraints, we won't prove this here. More importantly, we discuss why this theorem is a significant leap.

- It turns out the basis $\{\chi_\alpha\}_{\alpha \in A}$ of irreducible representations plays the same role as T that we had earlier for $L([-\pi, \pi])$.

The Fourier Series on a General Group!

Finally, after all this theory, we can define the Fourier series of a *class* function $f \in \mathcal{C}_G$!

Definition

Let $f \in \mathcal{C}_G$. Then, we define the Fourier series of f as follows

$$f(g) = \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle \chi(g) \text{ for all } g \in G,$$

where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of the group G.

The Fourier Series on a General Group!

Finally, after all this theory, we can define the Fourier series of a *class* function $f \in \mathcal{C}_G$!

Definition

Let $f \in \mathcal{C}_G$. Then, we define the Fourier series of f as follows

$$f(g) = \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle \chi(g) \text{ for all } g \in G,$$

where $\mathrm{Irr}(G)$ denotes the set of irreducible characters of the group G.

Remark

Notice the similarity with the Fourier series of a real function—again, this is just a **change of basis formula**!

$$\{f(g_k)\}_{k=1}^{n} \to \{\langle f, \chi_i \rangle\}_{i=1}^{n}.$$

The Fourier Series on a General Group! An Example

Example

Let's apply this to a **finite Abelian group**, say $\mathbb{Z}/N\mathbb{Z}$.

The Fourier Series on a General Group! An Example

Example

Let's apply this to a **finite Abelian group**, say $\mathbb{Z}/N\mathbb{Z}$.

- First, we'll have exactly N irreducible characters, and since $\mathbb{Z}_N$ is abelian, they'll be functions $\chi : \mathbb{Z}_N \to \mathbb{C}$ such that $\chi(x+y) = \chi(x)\chi(y)$ for all x and y in $\mathbb{Z}_N$.
- This naturally hints at the **exponential function**: we literally have $e^{a+b} = e^a e^b$ for real a and b, as we were taught so long ago.
- Hopefully then, it shouldn't be too surprising that $\chi_k(x) = e^{2\pi i k x/N}$ for $k = 0, \cdots, N-1$ are the irreducible characters for $\mathbb{Z}_N$—this matches up with our experience in $\mathbb{R}$!

The Fourier Series on a General Group! An Example

Example

Let's apply this to a **finite abelian group**, say $\mathbb{Z}/N\mathbb{Z}$.
As before, the Fourier series of f will be given by

$$f(x) = \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle \chi(x),$$

The Fourier Series on a General Group! An Example

Example

Let's apply this to a **finite abelian group**, say $\mathbb{Z}/N\mathbb{Z}$.
As before, the Fourier series of f will be given by

$$f(x) = \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle \chi(x),$$

which, in this case, becomes

$$f(x) = \sum_{k=0}^{N-1} \langle f, e^{2\pi ikx/N} \rangle e^{2\pi ikx/N} = \sum_{k=0}^{N-1} X_k e^{2\pi ikx/N},$$

where

$$X_k = \frac{1}{|G|} \sum_{x \in \mathbb{Z}_N} f(x)\overline{\chi_k(x)} = \frac{1}{N} \sum_{n=0}^{N-1} f(n) e^{-2\pi ikn/N}.$$

The Discrete Fourier Transform

Remark

Again, another way to think about it is simply as a **change of basis linear transformation** $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ given by

The Discrete Fourier Transform

Remark

Again, another way to think about it is simply as a **change of basis linear transformation** $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}\big((x_0, \cdots, x_{N-1})\big) = (X_0, \cdots, X_{N-1}) \text{ where } X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-2\pi i n k/N}.$$

The Discrete Fourier Transform

Remark

Again, another way to think about it is simply as a **change of basis linear transformation** $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}\big((x_0, \cdots, x_{N-1})\big) = (X_0, \cdots, X_{N-1}) \text{ where } X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-2\pi i n k/N}.$$

This is the **Discrete Fourier Transform!**

What Next?

Further Steps

- At the moment, this works **only for class functions**, not an arbitrary function in $\mathbb{C}[G]$.
  - **Schur's first orthogonality relation**: consider the **matrix coefficients** of irreducible representations.

$$\rho(g) = \begin{pmatrix} \rho_{11}(g) & \rho_{12}(g) & \cdots & \rho_{1n}(g) \\ \rho_{21}(g) & \rho_{22}(g) & \cdots & \rho_{2n}(g) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1}(g) & \rho_{n2}(g) & \cdots & \rho_{nn}(g) \end{pmatrix}$$

- This theory only works for **finite groups** G.
  - Remember the inner product on $\mathbb{C}[G] : \langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$ for all $f_1, f_2 \in \mathbb{C}[G]$.
  - The **Haar measure**, which is a measure on the group G, that allows us to re-define the inner product suitably for infinite groups,

$$\langle f_1, f_2 \rangle = \int_G f_1(x) f_2(x) d\mu(x).$$

  - The **Peter-Weyl** theorem generalizes this to all **compact (topological) groups**, which includes $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$, the circle group.
- Feel free to see my paper for more!

Thank You for Your Attention!