Stokes Theorem

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July 9, 2024

Intro

- Stokes in multivariable calculus
- differential forms
- Generalized Stokes theorem

Multivariable calculus

- Stokes Theorem:

$$\oint_C Fdr = \iint_S \nabla \times fdS$$

Differential Forms

- What is df in a differential equation?
- Why do we need it in calculus?

Differential Forms

k-form: $\alpha = \sum_I f_I e_I^*$, $I = i_1, i_2, ..., i_n$
exterior derivative: $f : U \to R$is

$$df := \sum_i \frac{\partial f}{\partial e_i} e_i^*$$

Topology

- what is manifold: topological space that locally resembles Euclidean space near each point
- diffeomorphism: a map that maps one differentiable manifold to another
- Let n, n_1 be two non-negative integers with $n \leq n_1$. An subset U of R^{n_1} is a n-manifold if for every point $p \in U$, there exists a neighborhood V of p in R^{n_1}, an open subset $X \subseteq R^n$ and a diffeomorephism $f : X \to U \cap V$.
- Partition of Unity

Pullback,differential form

Suppose $w_1, w_2, ..., w_n$ is a basis of U and $v_1, v_2, ..., v_m$ is a basis of V. Projection principle tells us, for a vector $u \in U, f(u) = f_1(u)w_1 + f_2(u)w_2 + ... + f_n(u)w_n$. A pullback of a k-form α is:

$$f^* \circ \alpha = \sum_{I \subseteq 1,...,m} (f_I \alpha)(D\alpha_{i_1} \wedge D\alpha_{i_2} \wedge ... \wedge D\alpha_{i_k})$$

This is a way to understand the differential k-form.

Manifold

chart: a pair (U, φ) where:

- U is an open subset of M.
- $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism, meaning φ is a continuous bijection with a continuous inverse.

integration of forms on manifolds:

$$\sum_{i=1}^{\infty} \int_X \rho_i \alpha$$

Stokes' Theorem

- $$\int_X d\alpha = \int_{\partial X} \alpha$$
- How to prove?

proof

1. If X is a half space H^n:
   $\int_{H^n} d\alpha = \sum_{i=1}^{n}(-1)^{i-1}\int_0^\infty \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\partial \alpha_i}{\partial x_i}(x) dx_1 \dots dx_n$
   Rearrange the order, for $i \neq n$, it equals
   $\sum_{i=1}^{n}(-1)^{i-1}\int_0^\infty \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \alpha_i(x)|_{x=-\infty}^{\infty} dx_1 \dots dx_n = 0$
   for $i = n$, $\sum_{i=1}^{n}(-1)^{i-1}\int_0^\infty \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\partial \alpha_n}{\partial x_n}(x) dx_n dx_1 \dots dx_{n-1}$
2. For arbitrary $(n-1)$ - manifold, choose a cover of supp α by finitely many oriented (generalized) coordinate charts (U_i, ϕ_i), and choose a subordinate partition of unity ϕ_i,we have

$$
\begin{aligned}
\int_{\partial X} \alpha &= \sum_i \int_{\partial X} \phi_i \alpha \\
&= \sum_i \int_{\partial X} d\phi_i \wedge \alpha + \phi_i \alpha \\
&= \int_X d(\sum_i \phi_i) \wedge \alpha + \int_X (\sum_i \phi_i) d\alpha \\
&= \int_X d\alpha
\end{aligned} \tag{1}
$$

Application

1. Math: Green's Theorem:

$$\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \int_{\partial D} PdxQdy$$

2. Physics: Maxwell-Faraday equation:

$$\nabla \times E = -\frac{\partial B(t)}{\partial t}$$

Thank you!