The Basel Problem and its Consequences

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Outline

What is the Basel Problem?

The Basel Problem was first introduced by Italian mathematician Pietro Mengoli in 1644. The problem asked to find the numerical value of:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

History of the Basel Problem

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- Around 1721, Goldbach estimated that the sum was between $\frac{41}{35}$ and $\frac{5}{3}$.

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- In 1734, Euler finally proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Euler would continue to provide proofs to the Basel problem, with one proposed in 1745 and another proposed in 1755.

Euler's Key Observation

To begin, Euler made the important insight that the coefficients of the infinite polynomial expression of $\sin x$ must be equal to the coefficients of the Maclaurin series of $\sin x$ for equal powers of x.

Infinite Polynomial Expression of $\sin x$

Fundamental Theorem of Algebra

If P(x) is a nonconstant polynomial with complex coefficients, then P(x) has at least one complex root.

Remark

This theorem implies that any polynomial of degree n with complex coefficients has n complex roots, with multiplicity included.

With similar insight to the Fundamental Theorem of Algebra, Euler then claimed that this must also be true for an infinite polynomial.

Infinite Polynomial Expression of $\sin x$

Therefore, understanding that the roots of $\sin x$ are $0, \pi, 2\pi, \ldots$, the infinite polynomial expression of $\sin x$ is

$$\sin x = lx(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots$$

In order to get a more accurate polynomial expression of $\sin x$, we must find l.

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$$1 = I(x - \pi)(x + \pi)(x - 2\pi)(x + 2\pi)\ldots.$$

Then, solving for I, we have

$$I = \frac{1}{(-\pi^2)(-4\pi^2)(-9\pi^2)}\cdots$$

Infinite Polynomial Expression of $\sin x$

Then substituting I into our original polynomial expression of $\sin x$ and heavily simplifying our expression, we get

$$\sin x = x \prod_{n=1}^{\infty} (1 - \frac{x^2}{(n\pi)^2}).$$

Therefore, we have found the infinite polynomial expression for $\sin x$. We can also manipulate this expression to obtain

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} (1 - \frac{x^2}{(n\pi)^2}).$$

Comparing Coefficients

We can now begin comparing coefficients of the same power term between the infinite polynomial expression of $\sin x$ and the Maclaurin series of $\sin x$.

Maclaurin Series of $\sin x$

The Maclaurin series of $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Comparing Coefficients

Looking at the coefficients of the x^2 terms of the infinite polynomial equivalent of $\frac{\sin x}{x}$, we see that

$$-(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots) = (\frac{-1}{\pi^2})(1 + \frac{1}{4} + \frac{1}{9} + \dots) = (\frac{-1}{\pi^2})\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

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Additionally, the coefficient of the x^2 term of the Maclaurin series of $\frac{\sin x}{x}$ is $-\frac{1}{6}$.Therefore, equating the two coefficients together, we find

$$\frac{-1}{6} = (\frac{-1}{\pi^2})\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

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Thus,

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

and we have found the solution to the Basel problem through Euler's methods.

Summary

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- Find the Maclaurin series of $\sin x$.
- Compare the coefficients of the x^2 terms.
- Solve for $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

A Secondary Euler Proof

This is Euler's second out of three proofs to the Basel Problem. The difference is that this proof uses L'Hôpital's rule and Euler's formula.

Key Theorems and Formulas

L'Hôpital's Rule: $\frac{0}{0}$ Case

Assume f and g are continuous functions which are defined on an interval containing c. Also, assume that f and g are differentiable on this interval, with the exception of point c. Then, if $f(c) = 0$ and $g(c) = 0$, then

$$\textit{if} \lim_{x \to c} \frac{f'(x)}{g'(x)} = L \text{ then } \lim_{x \to c} \frac{f(x)}{g(x)} = L$$

Euler's Formula

For any x and for $i = \sqrt{-1}$,

$$e^{ix} = \cos x + i \sin x.$$

Remark

From Euler's Formula, we can derive the following identities:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2}.$$

A Secondary Euler Proof

We know that the $\sin x$ can be expressed as this polynomial:

$$\sin x = x(1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})\ldots$$

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If we substitute $x = \pi t$ into the above equation, we get

$$\sin(\pi t) = \pi t(1 - t)(1 + t)(\frac{2 - t}{2})(\frac{2 + t}{2}) \ldots$$

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Combining like terms, we get

$$\sin(\pi t) = \pi t(1 - t^2)(\frac{4 - t^2}{4})(\frac{9 - t^2}{9}) \dots$$

A Secondary Euler Proof

Then, taking the natural logarithm of both sides and using the properties of the natural logarithm gives us

$$\ln(\sin(\pi t)) = \ln(\pi) + \ln(t) + \ln(1 - t^2) + \ln(4 - t^2) - \ln(4) + \dots$$

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We then differentiate with respect to t:

$$\frac{\pi \cos(\pi t)}{\sin(\pi t)} = \frac{1}{t} - \frac{2t}{1 - t^2} - \frac{2t}{4 - t^2} - \dots$$

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Then, subtracting $\frac{1}{t}$ on both sides and multiplying both sides by $\frac{-1}{2t}$ obtains

$$\frac{1}{1 - t^2} + \frac{1}{4 - t^2} + \frac{1}{9 - t^2} + \dots = \frac{1}{2t^2} - \frac{\pi \cos(\pi t)}{2t \sin(\pi t)}.$$

A Secondary Euler Proof

Substituting $t = -ix$, we get

$$\frac{1}{1+x^2} + \frac{1}{4+x^2} + \frac{1}{9+x^2} + \ldots = \frac{\pi \cos(-i\pi x)}{2ix \sin(-i\pi x)} - \frac{1}{2x^2}.$$

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From Euler's formula, we then know

$$\frac{\cos(y)}{\sin(y)} = \frac{\frac{1}{2}(e^{iy}+e^{-iy})}{\frac{1}{2i}(e^{iy}-e^{-iy})} = \frac{i(e^{2iy}+1)}{e^{2iy}-1}.$$

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We then substitute $y = -i\pi x$ into the above equation to get

$$\frac{\pi \cos(-i\pi x)}{2ix \sin(-i\pi x)} = \frac{\pi}{2x} + \frac{\pi}{x(e^{2\pi x} - 1)}.$$

A Secondary Euler Proof

We then substitute $\frac{\pi}{2x} + \frac{\pi}{x(e^{2\pi x}-1)}$ for $\frac{\pi\cos(-i\pi x)}{2ix\sin(-i\pi x)}$ into our original equation to get

$$\frac{1}{1+x^2} + \frac{1}{4+x^2} + \frac{1}{9+x^2} + \ldots = \frac{\pi x e^{2\pi x} - e^{2\pi x} + \pi x + 1}{2x^2 e^{2\pi x} - 2x^2}.$$

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We then take the limit as x approaches zero to both sides, which results in

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{0}{0}.$$

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$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{0}{0}.$$

Therefore, we will now use L'Hôpital's Rule on the former expression of the right-hand side.

A Secondary Euler Proof

We apply L'Hôpital's Rule's numerous times and simplify

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{x \to 0} \left(\frac{\pi x e^{2\pi x} - e^{2\pi x} + \pi x + 1}{2x^2 e^{2\pi x} - 2x^2} \right)$$

$$= \lim_{x \to 0} \left(\frac{\pi - \pi e^{2\pi x} + 2\pi^2 e^{2\pi x}}{4x e^{2\pi x} + 4\pi x^2 e^{2\pi x} - 4x} \right)$$

$$\cdots$$

$$= \frac{\pi^3}{4\pi + 2\pi}$$

$$= \frac{\pi^2}{6}.$$

Therefore, we have now found the precise sum for the Basel problem as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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- Use Euler's Formula to make a substitution.
- Use L'Hôpital's Rule.

The Wallis Product

The Wallis Product was first discovered by English mathematician John Wallis in 1655.

The Wallis Product

The Wallis Product is

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots .$$

Although the Wallis product was discovered before Euler's first proof to the Basel problem, Euler's polynomial for $\sin x$ yields the Wallis product when we substitute $x = \frac{\pi}{2}$. The Wallis product is important because it represents π through the ratio and products of natural numbers.

The Zeta Function

The Zeta Function

We define the Riemann zeta function to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for all $s \in \mathbb{C}$ with $\Re(s) > 1$.

The zeta function is an extremely important topic of analytic number theory, and it's consequences are still extremely important today such as it's relation to the prime number theorem and the Riemann hypothesis.

Euler's Product

Euler's Product

We define Euler's product as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \ prime} \frac{1}{1 - p^{-s}}.$$

Euler's work on the zeta function also lead to the conclusion that the zeta function can be expressed as an infinite product of the prime numbers. Euler's product is important because it is essential to the German mathematician Bernhard Riemann's work.

The Sum of the Reciprocal of Primes is Divergent

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The sum of the reciprocal of all prime numbers diverges which can be represented as

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{p_k} = \infty.$$

Euler's work on the zeta function and prime numbers also lead to the conclusion that the sum of the reciprocal of primes is divergent. This is important because it strengthens Euclid's proof that there are an infinite number of primes and also was useful to show that the series of the reciprocals of primes exhibits log growth.

Euler's Derivation of the General Formula of $\zeta(2n)$

Formula for $\zeta(2n)$

The general formula for $\zeta(2n)$ is given by

$$\zeta(2n) = \frac{1}{2}(-1)^{n+1}\frac{B_{2n}}{(2n)!}(2\pi)^{2n}$$

Bernoulli numbers

Bernoulli number, represented as B_n can be defined as the coefficients of the exponential function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}$$

Euler's continued work on the zeta function lead to his general formula for $\zeta(2n)$ and the definition of the Bernoulli numbers.

Thank you!

Thank you for your time!

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