The Basel Problem and its Consequences

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Abstract

We introduce the Basel Problem, a key problem that has consequences in mathematical analysis and number theory, and present we Bernoulli's work and two of Euler's proofs to the problem. Then, we present corollaries and consequences to Euler's proof such as the Wallis product and the Riemann zeta function. We then conclude with a few equations and theorems relevant to analytic number theory such as Euler's product and the divergence of the reciprocal of primes.

1 Introduction

The Basel Problem was first introduced by Italian mathematician Pietro Mengoli in 1644. The problem asked to find the numerical value of [10]

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

In 1655, Wallis estimated the solution to three decimal places. Around thirty years later, in 1689, Jakob Bernoulli proved that the infinite sum must be less than two. In 1721, Daniel and Johann Bernoulli proposed that the sum is around $\frac{8}{5}$. Around the same time, Goldbach estimated that the sum was between $\frac{41}{35}$ and $\frac{5}{3}$ [7].

Although Leibniz and Wallis were unable to provide solutions to the Basel problem, they made important observations, which can be seen in Euler's proof. Wallis's product formula for π which will be explored later in the paper, and Leibniz's proof that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}=\frac{\pi}{4}$ [3] based on the properties of trigonometric power series provided important connections on the behavior of power series, infinite sums and products, and π. Despite this progress, the problem continued to stump mathematicians until 1734, when Euler first solved the problem, using the properties of the power series of the sine function and techniques of calculus. Euler would continue to provide proof of the Basel problem, with one proposed in 1745 and another proposed in 1755 [7].

In this paper, we will briefly start with Bernoulli's approximation of the Basel problem. Then, we will explore Euler's solutions to the Basel problem. After these solutions, we will present a result to the problem, which is the Wallis product and the Riemann Zeta Function. We then conclude with a few equations and theorems relevant to analytic number theory such as the Euler's product and the divergence of the reciprocal of primes.

2 Bernoulli's Approximation

A key observation to the upper limit of the Basel problem was made by Bernoulli who showed that the sum must be less than or equal to two [7]. This observation was found through the inequality that

$$n(n+1) \leq 2n^2.$$

From this inequality, Bernoulli showed that

$$\frac{1}{\frac{n(n+1)}{2}} \geq \frac{1}{n^2}.$$

It's then clear that $\sum_{n=1}^{\infty} \frac{1}{\frac{n(n+1)}{2}}$ converges to two as

$$\sum_{n=1}^{\infty} \left(\frac{1}{\frac{n(n+1)}{2}}\right) = 2 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right),$$

where

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right).$$

Therefore,

$$\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

And so,

$$\sum_{n=1}^{\infty} \left(\frac{1}{\frac{n(n+1)}{2}}\right) = 2.$$

Finally by the comparison test for series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2.$$

Although this approximation may seem simple, Bernoulli's estimation provided a valuable upper limit to the Basel problem.

3 Euler's First Proof

In order to solve the Basel problem, Euler made the astute observation that one can compare the coefficients of the Maclaurin series for $\sin x$ to the coefficients of a polynomial in which the polynomial is constructed using the zeroes of $\sin x$. From this observation, Euler understood that the coefficients of the polynomial and the Maclaurin series for $\sin x$ must be the same for terms with the same power of x.

The Macularin series for sin(x) is as follows [16]

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

Euler understood that any finite polynomial could be written as the product of its linear factors determined by its roots. This observation is similar to the implications from the Fundamental Theorem of Algebra.

Theorem (Fundamental Theorem of Algebra)**.** *If P(x) is a non-constant polynomial with complex coefficients, then P(x) has at least one complex root. [12]*

Remark 1. *This Fundamental Theorem of Algebra also implies that any polynomial of degree n with complex coefficients has n complex roots, with multiplicity included. [12]*

Euler then used this observation to claim that the same can be said for an infinite polynomial. That is, Euler claimed that one can also write an infinite degree polynomial as a product of linear factors determined by its roots. However, this observation is not true for all polynomials. Therefore, using this observation and understanding that the roots of $\sin(x)$ are $\pi n, n \in \mathbb{Z}$. Therefore we can write the polynomial based on the roots of $\sin(x)$ as

$$P(x) = Ix(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)(x-4\pi)(x+4\pi)\dots$$

In order to find the leading coefficients, I, Euler noticed that [5]

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Therefore, substituting $P(x)$ into the limit, we see that

$$1 = \lim_{x\to 0} \frac{Ix(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\dots}{x}.$$

Therefore,

$$1 = I(x-\pi)(1+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots.$$

Thus,

$$I = \frac{1}{(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots}$$

We then substitute zero for x to get

$$I = \frac{1}{(-(\pi^2))(-(4\pi^2))(-(9\pi^2))\ldots}$$

Therefore,

$$\begin{aligned}
\sin x &= Ix(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots \\
&= \frac{x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots}{(-(\pi^2))(-(4\pi^2))(-(9\pi^2))\ldots} \\
&= \frac{x(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)\ldots}{(-(\pi^2))(-(4\pi^2))(-(9\pi^2)\ldots} \\
&= \frac{x(\pi^2-x^2)(4\pi^2-x^2)(9\pi^2-x^2)\ldots}{(\pi^2)(4\pi^2)(9\pi^2)\ldots} \\
&= x\cdot(\frac{\pi^2-x^2}{\pi^2})\cdot(\frac{4\pi^2-x^2}{4\pi^2})\cdot(\frac{9\pi^2-x^2}{9\pi^2})\ldots \\
&= x\cdot(1-\frac{x^2}{\pi^2})\cdot(1-\frac{x^2}{4\pi^2})\cdot(1-\frac{x^2}{9\pi^2})\ldots \\
&= x\prod_{n=1}^{\infty}(1-\frac{x^2}{(n\pi)^2}).
\end{aligned}$$

Therefore, Euler showed out that the polynomial equivalent to $\sin x$ is

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right)$$

Therefore, $\frac{\sin x}{x}$ is equivalent to

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right).$$

In order to solve the Basel Problem, we need to compare the coefficients of the x^2 terms of the polynomial equivalent $\frac{\sin x}{x}$ to the coefficient of the x^2 terms for the Maclaurin series of

$\sin x$. For the coefficients of the x^2 terms of the polynomial equivalent $\frac{\sin x}{x}$, we see that

$$-(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\ldots)=(\frac{-1}{\pi^2})(1+\frac{1}{4}+\frac{1}{9}+\ldots)=(\frac{-1}{\pi^2})\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Moreover, we know that the coefficient of the x^2 terms of the Maclaurin series for $\sin x$ is $\frac{-1}{6}$. Therefore,

$$\frac{-1}{6}=(\frac{-1}{\pi^2})\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Therefore, by doing a simple manipulation, we see that

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2},$$

which shows us how Euler found the precise sum to the Basel problem.

4 Euler's Proof Based on L'Hôpital's Rule and Euler's Formula

In this proof, Euler used the techniques of calculus such as L'Hôpital's rule and differentiation, along with an astute change of variables and again writing $\sin x$ as a product of its linear factors determined by its roots to solve this problem.

We know that the $\sin x$ can be expressed through this polynomial:

$$\sin x = x(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})\ldots$$

If we substitute $x=\pi t$ into the above equation, we get

$$\sin(\pi t)=\pi t(1-t)(1+t)(\frac{2-t}{2})(\frac{2+t}{2})\ldots$$

Combining like terms, we get

$$\sin(\pi t)=\pi t(1-t^2)(\frac{4-t^2}{4})(\frac{9-t^2}{9})\ldots$$

Then, taking the natural logarithm of both sides and using the properties of the natural logarithm gives us

$$\ln(\sin(\pi t)) = \ln(\pi) + \ln(t) + \ln\big(1 - t^2\big) + \ln\big(4 - t^2\big) - \ln(4) + \dots$$

We then differentiate with respect to t and get

$$\frac{\pi\cos(\pi t)}{\sin(\pi t)} = \frac{1}{t} - \frac{2t}{1-t^2} - \frac{2t}{4-t^2} - \dots$$

. We then subtract $\frac{1}{t}$ on both sides and multiply both sides by $\frac{-1}{2t}$ to obtain

$$\frac{1}{1-t^2} + \frac{1}{4-t^2} + \frac{1}{9-t^2} + \dots = \frac{1}{2t^2} - \frac{\pi\cos(\pi t)}{2t\sin(\pi t)}.$$

We then substitute $t = -ix$ to get

$$\frac{1}{1+x^2} + \frac{1}{4+x^2} + \frac{1}{9+x^2} + \dots = \frac{\pi\cos(-i\pi x)}{2ix\sin(-i\pi x)} - \frac{1}{2x^2}.$$

Then, we use Euler's formula which states that for any x and for $i = \sqrt{-1}$ [6],

$$e^{ix} = \cos x + i\sin x$$

to derive the following identities [6]

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2}.$$

From Euler's formula, we then know

$$\frac{\cos(y)}{\sin(y)} = \frac{\frac{1}{2}(e^{iy} + e^{-iy})}{\frac{1}{2i}(e^{iy} - e^{-iy})} = \frac{i(e^{2iy}+1)}{e^{2iy}-1}$$

We then substitute $y = -i\pi x$ into the above equation to get

$$\begin{aligned}\frac{\pi\cos(-i\pi x)}{2ix\sin(-i\pi x)} &= \frac{\pi}{2ix}\cdot\frac{i(e^{2\pi x}+1)}{e^{2\pi x}-1}\\ &= \frac{\pi}{2x}\cdot\frac{(e^{2\pi x}-1)+2}{e^{2\pi x}-1}\\ &= \frac{\pi}{2x}+\frac{\pi}{x(e^{2\pi x}-1)}\end{aligned}$$

We then substitute $\frac{\pi}{2x}+\frac{\pi}{x(e^{2\pi x}-1)}$ for $\frac{\pi\cos(-i\pi x)}{2ix\sin(-i\pi x)}$ into our original equation to get

$$\begin{aligned}\frac{1}{1+x^2}+\frac{1}{4+x^2}+\frac{1}{9+x^2}+\ldots &= \frac{-1}{2x^2}+\frac{\pi\cos(i\pi x)}{2ix\sin(-i\pi x)}\\ &= \frac{\pi x-1}{2x^2}+\frac{\pi}{x(e^{2\pi x}-1)}\\ &= \frac{\pi x e^{2\pi x}-e^{2\pi x}+\pi x+1}{2x^2e^{2\pi x}-2x^2}\end{aligned}$$

We then take the limit as x approaches zero to both sides. On the left hand side, we will apply Tannery's Theorem as we then take the limit as x approaches zero. We will also apply Tannery's Theorem to the left hand sides.

Theorem (Tannery's Theorem)**.** *Let* $S_n = \sum_{k=0}^{\infty} a_k(n)$*, and assume that* $\lim_{n\to\infty} a_k(n) = b_k$*. If* $|a_k(n)| \le M_k$ *and* $\sum_{k=0}^{\infty} M_k < \infty$*, then* $lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} b_k$ *[15].*

Applying Tannery's Theorem to the left hand side gives us

$$\lim_{x\to 0}\sum_{n=1}^{\infty}\frac{1}{n^2+x^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

After taking the limit as x approaches zero to both sides, we are then given the indeterminate form $\frac{0}{0}$ on the right hand side. Seeing this, we can now use L'Hôpital's Rule.

Theorem (L'Hôpital's Rule: $\frac{0}{0}$ Case)**.** *Assume f and g are continuous functions which are defined on an interval containing c. Also, assume that f and g are differentiable on this interval, with the exception of point c. Then, if $f(c) = 0$ and $g(c) = 0$, then*

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} = L$$

implies

$$\lim_{x\to c}\frac{f(x)}{g(x)} = L$$

[1].

We apply L'Hôpital's Rule's numerous times and simplify

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \lim_{x\to 0}\left(\frac{\pi x e^{2\pi x} - e^{2\pi x} + \pi x + 1}{2x^2 e^{2\pi x} - 2x^2}\right) \\ &= \lim_{x\to 0}\left(\frac{\pi - \pi e^{2\pi x} + 2\pi^2 e^{2\pi x}}{4xe^{2\pi x} + 4\pi x^2 e^{2\pi x} - 4x}\right) \\ &= \lim_{x\to 0}\left(\frac{\pi^3 x e^{2\pi x}}{e^{2\pi x} + 4\pi x e^{2\pi x} + 2\pi^2 x^2 e^{2\pi x} - 1}\right) \\ &= \lim_{x\to 0}\left(\frac{\pi^2(2\pi x + 1)}{4\pi^2 x^2 + 12\pi x + 6}\right) \\ &= \frac{\pi^3}{4\pi + 2\pi} \\ &= \frac{\pi^2}{6}.\end{aligned}$$

Therefore, we have found the precise sum for the Basel problem which is

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

5 Consequences of Euler's Proof of the Basel Problem

5.1 The Wallis Product

The Wallis Product was first discovered by English mathematician John Wallis in 1655. The Wallis Product [9] is

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots.$$

Although the Wallis product was discovered before Euler's first proof to the Basel problem, Euler's polynomial for $\sin x$ also yields us the Wallis product when we substitute $x = \frac{\pi}{2}$. We will also explore the more modern and common proof to the Wallis product, and an alternative form of the Wallis product. The original proof begins with an application of the integration by parts formula.

Theorem (Integration by Parts)**.** *Let u and v be differentiable functions of x on an interval I containing a and b. Then $\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx$. [1]*

We first let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ and understand that $\sin^n x = \sin^{n-1} x \cdot \sin x$. We then let

let $u(x) = \sin^{n-1}(x)$ and $v(x) = -\cos(x)$ and apply integration by parts. This results in

$$\begin{aligned}
I(n) &= \int_0^{\frac{\pi}{2}} \sin^n x \, dx \\
&= -\sin^{n-1} x \cos x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)(n-1)\sin^{n-2} x \cos x \, dx \\
&= 0 + (n-1) \int_0^{\frac{\pi}{2}} \cos^2 x \sin^{n-2} x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \sin^{n-2} x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x \, dx \\
&= (n-1) I(n-2) - (n-1) I(n) \\
&= \frac{n-1}{n} I(n-2).
\end{aligned}$$

From this, then we know that

$$\frac{I(n)}{I(n-2)} = \frac{n-1}{n}.$$

Then we make two substitutions that will be relevant in future steps.

$$\begin{aligned}
I(2n) &= \frac{2n-1}{2n} I(2n-2) \\
I(2n+1) &= \frac{2n}{2n+1} I(2n-1).
\end{aligned}$$

We also substitute $n = 0, 1$ into $I(n)$ to get

$$\begin{aligned}
I(0) &= \int_0^{\frac{\pi}{2}} dx = x \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} \\
I(1) &= \int_0^{\frac{\pi}{2}} \sin x \, dx = (-\cos x) \Big|_0^{\frac{\pi}{2}} = 1.
\end{aligned}$$

We then separate the cases and calculate $I(2n)$ and $I(2n+1)$.

$$\begin{aligned}
I(2n) &= \frac{2n-1}{2n} I(2n-2) \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} I(2n-4) \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdot \dots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I(0) \\
&= \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k} \\
I(2n+1) &= \frac{2n}{2n+1} I(2n-1) \\
&= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-2} I(2n-3) \\
&= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \dots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} I(1) \\
&= \prod_{k=1}^{n} \frac{2k}{2k+1}.
\end{aligned}$$

By properties of the $\sin x$ function, we see that

$$0 \le \sin^{n+1} x \le \sin^n x \le 1 \text{ for } 0 \le x \le \frac{\pi}{2}.$$

Therefore,

$$0 \le \sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x.$$

Therefore, we know that

$$0 < \sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x.$$

Therefore,

$$0 < I_{2n+1} \le I_{2n} \le I_{2n-1}.$$

From the recurrence relation, we have that

$$\frac{I_{2n-1}}{I_{2n}+1} = \frac{2n+1}{2n}.$$

We then divide $0 < I_{2n+1} \le I_{2n} \le I_{2n-1}$ by I_{2n+1}

$$1 \le \frac{I_{2n}}{I_{2n+1}} \le \frac{2n-1}{2n+1} = \frac{2n+1}{2n}.$$

Then, we use the squeeze theorem.

Theorem (Squeeze Theorem)**.** *Assume that $g(x) \leq f(x) \leq h(x)$ for all x close to c but not equal to c. If $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} f(x) = L$ [1].*

Therefore,

$$\lim_{n \to \infty} \frac{I_{2n}}{I_{2n+1}} = 1.$$

From this, we know that

$$\lim_{n \to \infty} \frac{I_{2n}}{I_{2n+1}} = \frac{\pi}{2} \prod_{k=1}^{n} (\frac{2k-1}{2k} \cdot \frac{2k+1}{2k}) = 1.$$

Therefore, we have derived the Wallis product as

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} (\frac{2k}{2k-1} \cdot \frac{2k}{2k+1}) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

Since we have finished the original proof to the Wallis product, we can explore how the Wallis product can be derived from Euler's proof to the Basel problem.

We start with the polynomial for $\sin x$.

$$\sin x = x \prod_{n=1}^{\infty} (1 - \frac{x^2}{(nx)^2}).$$

We then solve for a common denominator and substitute $x = \frac{\pi}{2}$:

$$\sin x = x \prod_{n=1}^{\infty} (\frac{n^2 \pi^2 - x^2}{n^2 \pi^2})$$
$$\sin \frac{\pi}{2} = \frac{\pi}{2} \prod_{n=1}^{\infty} (\frac{n^2 \pi^2 - (\frac{\pi^2}{4})}{n^2 \pi^2}).$$

We then factor out and cancel π^2:

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} (\frac{\pi^2 (n^2 - \frac{1}{4})}{\pi^2 n^2})$$
$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} (\frac{(n^2 - \frac{1}{4})}{n^2}).$$

We then solve for a common denominator and simplify

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} (\frac{\frac{4n^2-1}{4}}{n^2})$$

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} (\frac{4n^2-1}{4n^2}).$$

We then multiply both sides by $\frac{\pi}{2}$ and take the reciprocal of each side:

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} (\frac{4n^2-1}{4n^2})$$

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} (\frac{4n^2}{4n^2-1}).$$

We then are factor the numerator and denominator:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} (\frac{(2n)(2n)}{(2n+1)(2n-1)}).$$

Therefore, we are finished as

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} (\frac{(2n)(2n)}{(2n+1)(2n-1)}) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

which is the Wallis Product. We will now show and derive the alternative form of the Wallis Product which is

$$\sqrt{\pi} = \lim_{n \to \infty} \frac{2^{2n}(n!)^2}{(2n!)\sqrt{n}.}$$

We will first state some simple formulas. First, the nth product of even numbers:

$$(2) \cdot (4) \cdot (6) \cdot (8) \cdot \ldots \cdot (2n) \cdot \ldots = 2^n n!.$$

Second, the nth product of odd numbers:

$$(1) \cdot (3) \cdot (7) \cdot (9) \cdot \ldots \cdot (2n+1) \cdot \ldots = \frac{(2n+1)!}{2^n n!}.$$

Using the Wallis product formula, we see that

$$\begin{aligned}\frac{\pi}{2} &= \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} \\ &= \lim_{n\to\infty} \big(\frac{2^n(n!)^2}{(\prod_{k=1}^{n}(2k-1))(\prod_{k=1}^{n}(2k+1))}\big)\end{aligned}$$

Substituting the above formulas for the nth product of odd numbers and even numbers, we see that

$$\begin{aligned}\frac{\pi}{2} &= \lim_{n\to\infty} \big(\frac{2^{2n}(n!)^2(2^n n!)(2^n n!(2n+1))}{((2n+1)!)^2}\big) \\ &= \lim_{n\to\infty} \big(\frac{2^{2n}(n!)^2}{(2n)!}\big)^2\big(\frac{2n}{2n(2n+1)}\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\pi &= \big(\lim_{n\to\infty} \big(\frac{2^{2n}(n!)^2}{(2n)!}\big)^2\big(\frac{1}{n}\big)\big)\big(\lim_{n\to\infty} \frac{2n}{2n+1}\big) \\ &= \big(\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}}\big)^2.\end{aligned}$$

Therefore

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2^2 n(n!)^2}{(2n)!\sqrt{n}},$$

which is the alternative form of the Wallis product.

5.2 The Zeta Function

The Zeta Function is an extremely important topic of analytic number theory, and it's consequences are still extremely important today such as it's relation to the prime number theorem and the Riemann hypothesis.

Definition (Zeta Function)**.** *We define the Riemann zeta function to be [14]*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for all $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$.

5.3 Euler's Product

The zeta function was first discovered by Euler around the same time in which Euler solved the Basel problem. Euler also described the zeta function in terms of an infinite product termed as Euler's product. Euler's product also played a pivotal role in Riemann's own work, and is said to be an important idea in the eventual proof of the Riemann hypothesis.

Definition (Euler product)**.** *We define Euler's product as [8]*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{pprime} \frac{1}{1-p^{-s}}.$$

We will now proof Euler's product in a method similar to Euler's original proof.

We first see that when we multiply $\zeta(s)$ by $\frac{1}{2^s}$, we have

$$\frac{1}{2^s}\big(\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots\big)$$
$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \dots$$

We then subtract the second equation from the original zeta function equation:

$$(1 - \frac{1}{2^s})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots$$

We then multiply the entire former equation by the next prime:

$$\frac{1}{3^s}(1 - \frac{1}{2^s})\zeta(s) = \frac{1}{3^s} + 1\frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \dots$$

We then subtract the former equation from the original zeta function equation:

$$(1 - \frac{1}{3^s})(1 - \frac{1}{2^s})\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \dots$$

We then repeat this process an infinite amount of times for $\frac{1}{p^s}$ where p is prime:

$$\dots(1 - \frac{1}{11^s})(1 - \frac{1}{7^s})(1 - \frac{1}{5^s})(1 - \frac{1}{3^s})(1 - \frac{1}{2^s})\zeta(s) = 1$$

We then write this as

$$\zeta(s) = \prod_{p\,prime} \frac{1}{1-p^{-s}},$$

which is Euler's product.

5.4 The Sum of the Reciprocal of Primes is Divergent

Theorem (The Sum of Reciprocal of Primes is Divergent)**.** *The sum of the reciprocal of all prime numbers diverges which can be represented as [11]*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{p_k}=\infty.$$

We will use Euler's product to prove this theorem. However, before we begin, we will restate a few useful expressions.

First, that

$$\sum_{k=0}^{\infty}\frac{1}{x^k}=\frac{1}{1-x^{-1}}$$

for $x>1$.

Secondly, the taylor expansion of $\ln x$ is

$$\ln\frac{1}{1-x^{-1}}=\sum_{k=0}^{\infty}\frac{1}{kx^k}$$

for $x>1$. Third, the prime factorization theorem.

Theorem (Prime Factorization Theorem)**.** *Let k be a positive integer. Then there exists prime numbers $p_1,p_2,p_3,...,p_n$ and integers $i_1,i_2,i_3,...,i_n$ such that $k=p_1^{i_1}p_2^{i_2}\dots p_n^{i_n}$.*

In order to prove this theorem, we will compare the harmonic series to the sum of the reciprocal of prime numbers. To compare the harmonic series to the sum of the reciprocal of prime numbers, we will create an upper bound on the partial sums of the harmonic series.

To begin, by the prime factorization theorem and the formula $\sum_{k=0}^{\infty}\frac{1}{x^k}=\frac{1}{1-x^{-1}}$, we find [2]

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k}&\le\prod_{k=1}^{n}\sum_{j=0}^{n}\frac{1}{p_k^j}\\&\le\prod_{k=1}^{n}\sum_{j=0}^{\infty}\frac{1}{p_k^j}\\&=\prod_{k=1}^{n}\frac{1}{1-p_k^{-1}}\end{aligned}$$

Therefore, understanding that ln is increasing, we have

$$\begin{aligned}
\ln \sum_{k=1}^{n} \frac{1}{k} &\leq \ln \prod_{k=1}^{n} \frac{1}{1-p_k^{-1}} \\
&= \sum_{k=1}^{n} \ln \frac{1}{1-p_k^{-1}} \\
&= \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{1}{jp_k^j} \\
&= \sum_{k=1}^{n} \frac{1}{p_k} + \sum_{k=1}^{n} \sum_{j=2}^{\infty} \frac{1}{jp_k^j} \\
&\leq \sum_{k=1}^{n} \frac{1}{p_k} + \sum_{k=1}^{n} \sum_{j=2}^{\infty} \frac{1}{jp_k^j} \\
&= \sum_{k=1}^{n} \frac{1}{p_k} + \sum_{k=1}^{n} \frac{p_k^{-2}}{1-p_k^{-1}} \\
&\leq \sum_{k=1}^{n} \frac{1}{p_k} + \sum_{i=2}^{\infty} \frac{1}{i^2-i} \\
&= \sum_{k=1}^{n} \frac{1}{p_k} + 1.
\end{aligned}$$

Since the harmonic series is divergent which means that left hand side will also be divergent, then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{p_k} = \infty.$$

Therefore, using Euler's product and a few other mathematical expressions, we have showed that the sum of the reciprocal of primes diverges.

5.5 Euler's derivation of the formula for $\zeta(2n)$

Another consequence of Euler's initial solution to the Basel Problem was his continued work on the zeta function, namely that $\zeta(4) = \frac{\pi^4}{90}$ and that $\zeta(6) = \frac{\pi^6}{945}$ using similar techniques to his original proof of $\zeta(2) = \frac{\pi^2}{6}$ [13], which eventually lead Euler to derive the general formula for $\zeta(2n)$.

Theorem (Formula for $\zeta(2n)$)**.** *The general formula for $\zeta(2n)$ is given by [4]*

$$\zeta(2n) = \frac{1}{2}(-1)^{n+1}\frac{B_{2n}}{(2n)!}(2\pi)^{2n}.$$

The symbol B_{2m} are Bernoulli numbers which are relevant in number theory and analysis, and which were also discovered by Euler when he derived the formula for $\zeta(2n)$.

Definition (Bernoulli numbers)**.** *Bernoulli numbers can be defined as the coefficients of the exponential function [4]*

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

In order to find the general formula for $\zeta(2n)$, we first began with the infinite polynomial expression of $\sin x$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}(1 - \frac{x^2}{(\pi n)^2}).$$

To find the general formula for $\zeta(2n)$, we must find some way to relate the infinite polynomial expression of $\sin x$ to cot. To begin, we recognize the relation

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\ln(\sin x)\big) = \frac{\cos x}{\sin x} = \cot x$$

Taking the natural logarithm of $\frac{\sin x}{x}$, we get

$$\begin{aligned}
\ln \frac{\sin x}{x} &= \ln\left(\prod_{n=1}^{\infty}(1 - \frac{x}{\pi x})^2\right) \\
\ln \sin x - \ln x &= \ln\left(\prod_{n=1}^{\infty}(1 - \frac{x}{\pi x})^2\right) \\
\ln \sin x &= \sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{(\pi x)^2}\right) + \ln x.
\end{aligned}$$

We then take the derivative of each side:

$$\begin{aligned}
\cot x &= \frac{1}{x} + \sum_{n=1}^{\infty}(\frac{2x}{x^2 - (\pi x)^2}) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}(\frac{1}{1 + \frac{x}{\pi n}} + \frac{1}{1 - \frac{x}{\pi n}}).
\end{aligned}$$

We then substitute the power series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ and $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$ into the above equation:

$$\begin{aligned}
\cot x &= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\big(\sum_{k=0}^{\infty}(-1)^k(\frac{x}{\pi n})^k - \sum_{k=0}^{\infty}(\frac{x}{\pi n})^k\big) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\big((\sum_{k=0}^{\infty}(\frac{x}{\pi n})^{2k} - (\frac{x}{\pi n})^{2k+1}\big) - \big(\sum_{k=0}^{\infty}(\frac{x}{\pi n})^{2k} - (\frac{x}{\pi n})^{2k+1})\big) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\big(2\sum_{k=0}^{\infty}(\frac{x}{\pi n})^{2k+1}\big).
\end{aligned}$$

Then, changing k to $k+1$, and changing the order of summation, we have

$$\begin{aligned}
\cot x &= \frac{1}{x} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\big(\frac{2x^{2k-1}}{(n\pi)^{2k}}\big) \\
&= \frac{1}{x} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{2x^{2k-1}}{\pi^{2k}} \cdot \frac{1}{n^{2k}} \\
&= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x^{2k-=1}}{(\pi)^2k}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} \\
&= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x^{2k-1}}{\pi^{2k}}\zeta(2n).
\end{aligned}$$

Now, we have a cotangent function written in terms of the zeta function. The next step is to find a cotangent function written in terms of the Bernoulli numbers. In order to so, we must state other equations involving the Bernoulli numbers. Firstly, from our definition of the Bernoulli numbers involving the exponential generating function, we know that

$$\frac{x}{e^x - 1} = B_0 + B_1 x + \frac{B_2 x^2}{2!} + \frac{B_3 x^3}{3!} + \dots$$

We then subtract $B_1 x$ to get

$$\begin{aligned}
f(x) &= \frac{x}{e^x - 1} - B_1 x \\
&= \frac{x}{e^x - 1} + \frac{x}{2} \\
&= \frac{2x + x(e^x - 1)}{2(e^x - 1)} \\
&= \frac{x(e^x + 1)}{2(e^x - 1)} \\
&= \frac{x(e^x + 1)}{2(e^x - 1)}\left(\frac{e^{\frac{-x}{2}}}{e^{\frac{-x}{2}}}\right) \\
&= \frac{x(e^{\frac{x}{2}} + e^{\frac{-x}{2}})}{2(e^{\frac{x}{2} - e^{\frac{-x}{2}}})}.
\end{aligned}$$

Therefore,

$$f(x) = \frac{x}{e^x - 1} - B_1 x = \frac{x(e^{\frac{x}{2}} + e^{\frac{-x}{2}})}{2(e^{\frac{x}{2}} - e^{\frac{-x}{2}})}.$$

In order to rewrite the this equation in terms of Bernoulli numbers, we will have to substitute the hyperbolic $\sinh x$ and $\cosh x$ curves into the right hand side of the equation.

The hyperbolic $\sinh x$ and $\cosh x$ curves are

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2}, \\
\cosh x &= \frac{e^x + e^{-x}}{2}.
\end{aligned}$$

Using these hyperbolic equations, we see

$$\begin{aligned}
\frac{e^{\frac{x}{2}} + e^{\frac{-x}{2}}}{e^{\frac{x}{2}} - e^{\frac{-x}{2}}} &= \frac{\frac{e^{\frac{x}{2}} + e^{\frac{-x}{2}}}{2}}{\frac{e^{\frac{x}{2}} - e^{\frac{-x}{2}}}{2}} \\
&= \frac{\cosh \frac{x}{2}}{\sinh \frac{x}{2}} \\
&= \coth \frac{x}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{x(e^{\frac{x}{2}} + e^{\frac{-x}{2}})}{2(e^{\frac{x}{2}} - e^{\frac{-x}{2}})} \\ &= \frac{x}{2}\coth\frac{x}{2}. \end{aligned}$$

Since $f(x)$ has no negative odd terms, we can write $\frac{x}{2}\coth\frac{x}{2}$ as

$$\frac{x}{2}\coth\frac{x}{2} = \sum_{n=0}^{\infty}\frac{B_{2n}x^{2n}}{(2n)!}.$$

From this, we can derive a relation between the hyperbolic $\coth x$ curve and the Bernoulli expression

$$\begin{aligned} \coth x &= \sum_{n=0}^{\infty}\frac{B_{2n}(2x)^{2n}}{x(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{2B_{2n}(2x)^{2n-1}}{(2n)!}. \end{aligned}$$

Substituting xi for x in the equation, we get

$$\cot x = \sum_{n=0}^{\infty}(-1)^n\frac{2B_{2n}(2x)^{2n-1}}{(2n)!}.$$

Now that we have two different expressions for $\cot x$, we can prove Euler's general formula for $\zeta(2n)$.

Firstly, we can rewrite the $\cot x$ expression involving the Bernoulli numbers as

$$\begin{aligned} \cot x &= \sum_{k=0}^{\infty}(-1)^k\frac{2B_{2k}(2x)^{2k-1}}{(2k)!} \\ &= \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2B_{2k}(2x)^{2k-1}}{(2n)!}. \end{aligned}$$

We then set this equation with the cotangent equation involving the zeta function equal to each other. Doing this, we get

$$\frac{1}{x}+\sum_{k=1}^{\infty}\frac{2x^{2k-1}}{\pi^{2k}}\zeta(2k)=\frac{1}{x}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2B_{2k}(2x)^{2k-1}}{(2k)!}$$
$$\sum_{k=1}^{\infty}\frac{2x^{2k-1}}{\pi^{2k}}\zeta(2n)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2B_{2k}(2x)^{2k-1}}{(2k)!}.$$

Comparing the coefficients of terms with x^{2k+1}, we get

$$\frac{2}{\pi^{2k}}\zeta(2k)=(-1)^{k-1}\frac{2^{2k}B_{2k}}{(2k)!}.$$

We then isolate $\zeta(2k)$ to get

$$\zeta(2k)=(-1)^{k-1}\frac{(2\pi)^{2k}}{(2k)!}B_{2k}.$$

Therefore, we have found a general formula for $\zeta(2n)$.

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