Voting Oddities: When The Polls Go Wrong

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What is Social Choice?

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Suppose that you and a group of friends are at the grocery store. Each of you likes a different type of fruit. Thus you each buy the one you prefer, accounting for the prices. But suppose that now you must decide who will be the next President or who receives a kidney donation. Clearly a market-based approach is not effective, so it becomes important to aggregate people's desires. This is known as preference aggregation.

A Mathematical Approach to Politics

We choose an axiomatic approach to the problem as mathematicians. This allows us to analyze consequences that may arise as a result. as an example, a run-off may be used to determine the presidential election in the United States. However, sometimes choosing a certain set of axioms may result in an undesirable or impossible outcome, which is what we will investigate.

Conventions

Let a group of individuals be denoted N and the corresponding set of axioms denoted X. N is composed of n elements or people with index i, and we also assume that $|N| \geq 2$ and $|X| \geq 3$.
This allows us to develop a partial order over N, denoted with the preference relation $\succeq$. This is a weak order, which means that if person i prefers outcome x over outcome y, then $x \succeq_i y$.

Conventions

We are able to split the notation $\succeq$ into $\succ$ and $\sim$ at times, where the former represents strict preference and the latter represents indifference. We also use ρ to represent the set of preferences so that $\rho = (\succeq_i)$ for $1 \geq i \geq n$.

Choice Representation

Finally, we define a choice function F as one that takes a preference set as input and returns a winner.

We also define a preference aggregation rule f as one that takes a preference set as input and returns a group preference relation $\succeq$.

Example

Let $N = \{1, 2\}$ and $X = \{x, y, z\}$. If we assume that 1 has preferences $x \succeq_1 y \succeq_1 z$ and 2 has preferences $y \succeq_2 x \succeq_2 z$, then $\rho = \{x \succeq_1 y \succeq_1 z, y \succeq_2 x \succeq_2 z\}$.
Take the method of Borda count, where a preference a person ranks in the ith place has value $n - i$. Then $f_B(\rho) = y \succ x \succ z$ and $F_B(\rho) = y$.

Arrow's Impossibility Theorem

Arrow's theorem presents four axioms that are then shown to be impossible to coexist.

Definition (Pareto Efficiency)

A social choice function is Pareto efficient if whenever $a \succeq_i b$ for all i, then $a \succeq b$. If everyone prefers one outcome over another, then society also prefers that outcome over the other.

Definition (Dictator)

A social choice function has a dictator if aggregated preferences $\succeq$ are equal to some individual's preferences $\succeq_i$.

Arrow's Impossibility Theorem

Definition (Independence of Irrelevant Alternatives)

A social choice function is IIA if the set of voters that prefers a to b is the same as the set of voters that prefers x to y, then society ranks a and b the same way they rank x and y. Mathematically, a social choice function being IIA states that if $i : a \succ_i b = i : x \succ_i y$ then $a \succ b$ if and only if $x \succ y$.

Definition (Transitivity)

A relation $\succeq$ over a set G is transitive if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

This implies that there cannot exist a cycle where $x \succ y, y \succ z, z \succ x$ simultaneously.

Universal Assumption

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Definition (Unrestricted Domain)

The domain of all functions and rules includes all possible orderings of the preference relations of individuals.

In other words, we cannot restrict the preference of any one individual. This is very important later in the talk.

Arrow's Impossibility Theorem

Now we are ready to state the theorem.

Theorem (Arrow)

With three or more alternatives, any aggregation rule satisfying unrestricted domain, Pareto efficiency, IIA and transitivity is a dictator.

This essentially states that there must exist a dictator, one who holds all decision making authority if a preference rule satisfies the above conditions.

Proof Strategy

We first define a limited dictator $n*$, and then prove that $n*$ is a genuine dictator.

Definition (Pivotal)

$n*$ is pivotal for alternatives a, b in ρ if $n*$ causes the rest of society to follow them if they prefer $a \succeq b$.

Thus if we can show there exists a pivotal individual, it follows that there must be a dictator as well.

May's Theorem

Another such impossibility theorem is May's Theorem. He lays out four additional axioms:

Definition (Anonymity)

Given a preference aggregation f, each voter is treated equally; we are able to interchange any two $\succeq_i$ and $\succeq_j$ in $f(\rho)$.

Definition (Neutrality)

The aggregation rule cannot favor any alternative over another.

May's Theorem

Definition (Positive Responsiveness)

The group should have a positive response to individual preferences. If $f(\rho)$ generates $x \succeq y$, changing an individual response from $y \succeq_i x$ to $x \succeq_i y$ changes the group preference from $x \sim y$ to $x \succ y$.

Definition (Majority Rule)

Majority rule f_M is a rule over a alternatives such that $x \succ y$ if more people prefer x to y than y to x and vice versa.

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Sen's Theorem

We are able to detach individual choices from the group, defining another axiom:

Theorem (Minimal Liberalism)

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This is essentially a case of Arrow where the dictator is reduced to a decentralized power.

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Theorem (Sen's Theorem)

No aggregation rule that produces transitive outcomes can simultaneously satisfy unrestricted domain, Pareto efficiency, and minimal liberalism.

Gibbard-Satterthwaite Theorem

While the last two theorems were modified versions of Arrow's theorem, they only considered preference aggregation rules f. Gibbard and Satterthwaite considered choice functions F instead. It is also more realistic as preferences are not assumed to be true, rather they are ballots that may be false.

Gibbard-Satterthwaite Theorem

Suppose that $\rho = (\succeq_i)$ for $1 \leq i \leq n$ is a true preference profile, but $\rho' = (\succeq_1, \succeq_2, \ldots, \succeq_i', \ldots, \succeq_n)$ in which person i's preference is false.

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It turns out that the proof of this is similar to the proof of Arrow, although we will not discuss that here.

Conclusion

Voting is certainly a commonplace event in the modern world. It is nice to theorize among ideal scenarios in order to gain a deeper understanding of the mechanism behind it. Impossibility is one nice consequences; the other is paradoxes, which are similar but differ in significant ways, more information which can be found in my paper.

Credits

Thank you!