VOTING ODDITIES: WHEN THE POLLS GO HAYWIRE

SAMARTH DAS

ABSTRACT. This paper discusses various voting impossibilities and paradoxes that arise within social and collective choice theory, a branch of applied mathematics to politics that uses algebraic structures for rigorousness.

1. What is Social Choice?

Everyone possesses certain desires. One may like to eat apples while another may like to consume bananas. But how can we choose to distribute these items accounting for the tastes? Consider the denomination of prices: at a grocery store, those who are willing to pay the listed price for apples will purchase apples, and those who are willing to pay the listed price for bananas will purchase bananas. Yet in many cases, prices are not effective. For instance, if we're trying to decide who should next receive a kidney donation or who the next President should be, using a market-based mechanism would attract much criticism. In these situations, the science of how to aggregate people's diverse wants becomes important. This is known as **preference aggregation**.

2. A Mathematical Approach to Politics

As mathematicians, we choose an axiomatic approach to the problem. Thus when assuming a certain set of axioms, we can then analyze the consequences that arise from them. The reason we choose such an approach is due to the fact that encompasses both the voting procedure and the goal of the system. An example is that of the modern presidential election in the United States, where a run-off is used to determine electoral votes from states. Sometimes a certain set results in an undesirable situation or an impossible outcome, a few of which we will discuss in this paper.

3. Notations

For such a topic it is necessary to develop rigorous notations and definitions so that it is possible to perform further analysis. Let a group of individuals be denoted N and the corresponding set of axioms denoted X. N is composed of n elements or people with index i, and we also assume that $|N| \geq 2$ and $|X| \geq 3$. This allows us to develop a partial order over N, denoted with the preference relation $\succeq$. This is a weak order, which means that if person i prefers outcome x over outcome y, then $x \succeq_i y$. Note that $x \succeq y$ and $y \succeq x$ implies $x \sim y$, because then x is liked the same as y. Thus cycles are not permitted, since it is easy to think of this like the $\geq$ relation on the number line.

We are able to split the notation $\succeq$ into $\succ$ and $\sim$ at times, where the former represents strict preference and the latter represents indifference.

We also use ρ to represent the set of preferences so that $\rho = (\succeq_i)$ for $1 \leq i \leq n$.

Date: July 15, 2024.

Definition 3.1. A binary relation $\succeq$ over a set G is complete if for all elements x and y in G, either $x \succeq y$ or $y \succeq x$ (or both).

Definition 3.2. A binary relation $\succeq$ over a set G is transitive if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Consider an example: Let $N = \{1, 2\}$ and $X = \{x, y, z\}$. If we assume that 1 has preferences $x \succeq_1 y \succeq_1 z$ and 2 has preferences $y \succeq_2 x \succeq_2 z$, then $\rho = \{x \succeq_1 y \succeq_1 z, y \succeq_2 x \succeq_2 z\}$.

Take the method of Borda count, where a preference a person ranks in the ith place has value $n - i$. Then $f_B(\rho) = y \succ x \succ z$ and $F_B(\rho) = y$.

4. Preferences

Let us introduce Arrow's Impossibility Theorem with a few definitions.

Definition 4.1 (Social choice Function)**.** A social choice function is a function that takes in individual preferences and outputs some aggregated preferences.

Definition 4.2 (Efficiency)**.** A social choice function is efficient if whenever $a \succeq_i b$ for all i, then $a \succeq b$. If everyone prefers one outcome over another, then society also prefers that outcome over the other.

Definition 4.3 (Neutrality)**.** A social choice function is neutral if whenever all individuals' preferences between a and b are the same as their preferences between x and y, then the social choice function's preference between a and b are the same as its preference between x and y. That is, if the set of voters that prefers a to b is the same as the set of voters that prefers x to y, then society ranks a and b the same way they rank x and y.

Definition 4.4 (Dictator)**.** A social choice function has a dictator if aggregated preferences $\succeq$ are equal to some individual's preferences $\succeq_i$.

Definition 4.5 (Utility Function)**.** Given preferences $\succeq$ over the set G, a utility function u from G to real numbers represents $\succeq$ if for any a,b in G, we have that a $\succeq$ b if and only if u(a) $\geq$ u(b)

Definition 4.6 (Monotonicity)**.** A function f is monotonic if $x \geq y$ implies $f(x) \geq f(y)$.

Definition 4.7 (Independence of Irrelevant Alternatives)**.** A social choice function is IIA if the set of voters that prefers a to b is the same as the set of voters that prefers x to y, then society ranks a and b the same way they rank x and y. Mathematically, a social choice function being IIA states that if $i : a \succ_i b = i : x \succ_i y$ then $a \succ b$ if and only if $x \succ y$.

Definition 4.8 (Transitivity)**.** A relation $\succeq$ over a set G is transitive if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

This implies that there cannot exist a cycle where $x \succ y, y \succ z, z \succ x$ simultaneously.

Then arrow's theorem tells us that

Theorem 4.9 (Arrow)**.** *With three or more alternatives, any aggregation rule satisfying unrestricted domain, Pareto efficiency, IIA and transitivity has a dictator.*

Proof. Let X be the collection of all finite subsets of V. In other words,

$$X = \{W \subseteq V \mid W \text{is finite}\}$$

. For each $W \in X$, let $X_W = \{U \in X \mid W \subseteq U\}$, and let $\mathcal{F}$ be the set of all $Y \subseteq X$ such that, for some $W \in X, X_W \subseteq Y$.

Note that it is trivial that $\mathcal{F}$ is a filter on X by simply checking the definition conditions listed in **Definition 4.10**. Thus we can immediately apply the Axiom of Choice to find an ultrafilter $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \mathcal{G}$.

For every $W \in X$, we can find a function $f_W : W \to d$ such that, for all $v, w \in W$, if $\{v, w\} \in E$, then $f_W(v) \neq f_W(w)$. In other words, we can find a chromatic coloring $f_W : W \to \{0, 1, \ldots, d-1\}$.

For $w \in V$ and $i < d$, let

$$X_{\{w,i\}} = W \in X \mid w \in W \text{ and } f_W(w) = i$$

. The characteristics of $\mathcal{G}$ as an ultrafilter guarantee the existence of a unique $i_w < d$ such that $X_\{w, i_w\} \in \mathcal{G}$. Now all we have to do is to show that this is unique to all the other i_w. Let us formalize this argument by defining a function $f : V \to d$ by letting $f(w) = i_w$ for all $w \in V$.

We claim that f is a chromatic coloring, i.e., for all $\{u, v\} \in E$, we have $f(u) \neq f(v)$.

Proof. Fix $\{u, v\} \in E$. Since $\mathcal{G}$ is an ultrafilter we can find a set W in $X_\{u, i_u\} \cap X_\{v, i_v\}$. Now we must have $u, v \in W, f_W(u) = i_u$, and $f_W(v) = i_v$. Since f_W is a chromatic coloring and $\{u, v\} \in E$, it follows that $i_u \neq i_v$ and hence $f(u) \neq f(v)$. ■

■

We assume that all preference rules and choice functions satisfy the following property:

Definition 4.10 (Unrestricted Domain)**.** The domain of all functions and rules includes all possible orderings of the preference relations of individuals.

In other words, we cannot restrict the preference of any one individual.

We first define a limited dictator $n*$, and then prove that $n*$ is a genuine dictator.

Definition 4.11 (Pivotal)**.** $n*$ is pivotal for alternatives a, b in ρ if $n*$ causes the rest of society to follow them if they prefer $a \succeq b$.

Thus if we can show there exists a pivotal individual, it follows that there must be a dictator as well.

As preferences are sufficiently diverse, there exist outcomes x, y, z such that $x \succ_1 z$ and $x \succ_2 z$. Without loss of generality, suppose that $x \succ_1 y$ and $y \succ_2 x$. By efficiency, it must be that $x \succ z$. If the social choice function outputs $x \succ y$, then $x \succ 1y, y \succ 2x, x \succ y$ as desired. If the social choice function outputs $y \succ x$, then $y \succ 2x, x \succ 1y, y \succ x$ as desired by swapping x and y in the problem statement.

Consider person two's preferences. If $a \succ_2 b$, then both individuals prefer a to b so $a \succ b$ by efficiency. If $b \succ 2a$, then $i : ab = 1 = i : xy$ so the social choice function being neutral gives that $a \succ b$ if and only if $x \succ y$. As $x \succ y$ is given, we must have $a \succ b$

We will show that $\succ$ is equivalent to $\succ_i$. By the preceding problem, for any other outcomes a, b if $a \succ_1 b$ then $a \succ_b$. Repeatedly applying this going down individual one's preference ordering gives the desired result.

As preferences are sufficiently rich and the social choice function is efficient,Lemma 4.4 gives that there exists outcomes x and y such that one individual prefers x to y, the other individual prefers y to x, and the social choice function outputs $x \succ y$. Let that individual be individual 1. With this setup, the hypotheses of Lemma 4.6 are satisfied so individual 1 is a dictator.

5. Voting Systems

An important yet difficult problem plaguing society for years is that of preference aggregation: Given a set of individual preferences over some set of different outcomes, how can we aggregate that into some democratic societal ranking over outcomes? Suppose you and your friends want to celebrate surviving a a zombie Apocalypse. There are many choices that need to be made: Where should you go? When should the celebration take place? In general, societies have always struggled with preference aggregation. Even very natural aggregation schemes fail to preserve the desired properties of (individual) preference orderings. Suppose there are three individuals, Alice, Bob, Carl, and three possible outcomes, x,y,z. To simplify notation, we write preferences in the notation of linear orders: We list outcomes in a way such that the first outcome is the most-preferred, the second outcome is the second most-preferred, and so on. Linear orders induce complete and transitive preferences. For instance, let Alice, Bob, and Carl's preferences be:

- If for some i we have $a \prec_i b$ instead of $a \succ_i b$, then the choice function fails to output a relation for (a, b), which means the output of preferences is not complete.
- Transitivity is guaranteed since if all individuals prefer a to b and b to c, then they all prefer a to c and the aggregated preferences for unanimity will include all of those.
- If all individuals have $a \succ_i b$, then unanimity guarantees $a \succ b$ by definition
- If all individuals have $a \succ_i b$ and $x \succ_i y$ the unanimity function guarantees that $a \succ b$ and $x \succ y$.

Suppose we have $a_1, a_2, \ldots, a_n$ as outcomes. If the choice function is efficient, then if $a_i \succ_1 a_j$, we have $a_i \succ a_j$, which is clearly a dictatorship. Similarly, if we have $a_i \succ a_j$, then we must have $a_i \succ_1 a_j$, otherwise the social outcome would be different from the only individual's preference.

Consider the example of a dictatorship with 1 as the dictator, then it follows that $x \succ y$ and $x \succ_1 y$ and $y \succ_2 x$. This can be denoted with the fact that if there are two political coandidates and there is a strong pereference towards one that they must hold a literal dictatorship.

Since 1 is a dictator, their interests align with the majority, and thus since society prefers $x \succ y$ for some events x and y, it implies that for some other relatiosn a and b the preference of 1 would be aligning with the rest of society, and thus $a \succ_1 b$ implies $a \succ b$.

Let the individuals be 1 and 2, then we have that although completeness allows $x \succ_1 y$ and $y \succ_2 x$, society prefers $x \succ y$, and since there are only two individuals 1's preference corresponds to that of society, so 1 is a dictator.

6. Other Theorems

First let's state some definitions and lemmas.

Note that indifference is a symmetric relationship because suppose $x \sim y$. Then, $x \succeq y$and $y \succeq x$. By simply moving statements around, $y \succeq x$ and $x \succeq y$. Thus, $y \sim x$.

Definition 6.1 (Positive Responsiveness)**.** The group should have a positive response to individual preferences. If $f(\rho)$ generates $x \succeq y$, changing an individual response from $y \succeq_i x$ to $x \succeq_i y$ changes the group preference from $x \sim y$ to $x \succ y$.

Definition 6.2 (Majority Rule)**.** Another such impossibility theorem is May's Theorem. He lays out four additional axioms:

Definition 6.3 (Anonymity)**.** Given a preference aggregation f, each voter is treated equally; we are able to interchange any two $\succeq_i$ and $\succeq_j$ in $f(\rho)$.

Definition 6.4 (Neutrality)**.** The aggregation rule cannot favor any alternative over another.

Majority rule f_M is a rule over a alternatives such that $x \succ y$ if more people prefer x to y than y to x and vice versa.

Theorem 6.5 (May's Theorem)**.** *The only preference aggregation rule that is anonymous, neutral and positively responsive is the method of majority rule.*

Essentially this is the case of Arrow's theorem with only two alternatives instead of three or more alternatives in ρ. We are able to detach individual choices from the group, defining another axiom:

Theorem 6.6 (Minimal Liberalism)**.** *There exists at least two individuals and two alternatives such that each person is free to decide among the alternatives without affecting each other or the group's choice.*

This is essentially a case of Arrow where the dictator is reduced to a decentralized power.

Theorem 6.7 (Sen's Theorem)**.** *No aggregation rule that produces transitive outcomes can simultaneously satisfy unrestricted domain, Pareto efficiency, and minimal liberalism.*

While the last two theorems were modified versions of Arrow's theorem, they only considered preference aggregation rules f. Gibbard and Satterthwaite considered choice functions F instead. It is also more realistic as preferences are not assumed to be true, rather they are ballots that may be false.

Definition 6.8 (Strategy-Proofness of a choice function)**.** a choice function F is strategy-proof if $F(\rho')$ can never generate an outcome that i prefers to the outcome generated by $F(\rho)$.

Essentially it states that no voter can benefit by falsifying their preferences.

Theorem 6.9 (Gibbard-Satterthwaite Theorem)**.** *With three or more alternatives and unrestricted domain, any choice function that is strategy-proof and Pareto efficient is dictatorial.*

Suppose that $\rho = (\succeq_i)$ for $1 \leq i \leq n$ is a true preference profile, but $\rho' = (\succeq_1, \succeq_2, \ldots, \succeq_i'$ $, \ldots, \succeq_n)$ in which person i's preference is false.

7. Paradoxes

Consider the following scenarios:

Is indifference a symmetric relationship? That is, if$x \sim y$, then is it necessarily the case that y x? Suppose there are finitely many sequence $a_1, a_2, \ldots, a_n$ such that

$$a_1 \succeq a_2 \succeq a_3, \succeq \cdots \succeq a_n \succeq a_1.$$

(there can't be infinitely many since then G would be infinite) Then by transitivity, each of these cycles is all the same element. Between each cycle, we will have some relation, and since there are infinitely many relations for different values, there is going to be a maximal element. If there is a sequence for which there is no maximal element, then that sequence must be infinite, contradicting G is finite.

$$G\epsilon(a_1, a_2, a_3, \ldots, a_n)$$

is a complete set such that

$$a_1 \succeq a_2 \succeq a_3, \succeq \cdots \succeq a_n \succeq a_1$$

as n approaches infinity. By the transitivity of the binary relationship, all of these elements are going to be the same element as the cycle length approaches infinity because there would be an infinitive amount of relationships between the elements, meaning there would be no maximal element.

Suppose $x \succ y$ and $y \succ z$. If $x \prec z$, then we have

$$x \succ y \succ z \succ x,$$

which means $x \succ x$, but this is impossible. Thus, we must have $x \succeq z$. However, if $x = z$, then we have $x \succ y$ and $y \succ x$, which is again impossible. Thus, we have $x \succ z$.

For the maximum number of weak preferences, it can be assumed that all the elements in the set are indifferent, which means that between two elements in the set of G, there are two weak preferences such as that

$$x \succeq y \; and \; y \succeq x$$

So between each element of the set, there would be two weak preferences, which means that overall there would be n(n-1) of weak preferences.

Since $u(a) \geq u(b)$ if and only if $a \succeq b$, we have a homomorphism between the range of u and G, and since the real numbers are complete and transitive with the relation $\geq$, the set G must be be complete and transitive with the relation $\succeq$.

There is a bijection between values of $g(x)$ and values of $f(g(x))$ since f is monotonic so it is one-to-one, and if $g(x) > g(y)$, that implies $f(g(x)) > f(g(y))$

$x \succ y, y \succ z, z \succ x$ but transitivity implies $x \succ z$ so that would mean $x \succ x$ which is impossible.

- Alice: $y \succ x \succ z$
- Bob: $z \succ y \succ x$
- Carl: $x \succ z \succ y$

$y \succ x, x \succ z, z \succ y$ for society

- Alice and Bob have $y \succ x$ so x cannot be a Condorcet winner;
- Bob and Carl have $z \succ y$ so y cannot be a Condorcet winner;
- Alice and Carl have $x \succ z$ so z cannot be a Condorcet winner.

Assume that the set of G contains elements of $a_1, a_2, \ldots, a_{n-1}, a_n$ where n is a finite positive integer, and that $a_1 \succ a_2 \succ a_3 \succ \ldots, \succ a_{n-1} \succ a_n \succ a_1$. This means that G contains a cycle, and a cycle means that no Condorcet winner exists in the set since there is another element that is preferred in society for every element.

This gives us the condition that there is no Condorcet winner if the set is a cycle.

However, when the set of G containing elements of $a_1, a_2, \ldots, a_{n-1}, a_n$ where n is a finite positive integer and that it is no a cycle, thus the set can be arranged in a fashion of $a_1, \ldots, a_n$ such that weakly more people prefer a_i to a_{i+1} for every $i = 1, 2, \ldots, n-1$. In this case, a_1 would be considered as a Condorcet winner since $a_1 \succeq$ every other element.

This gives us the condition that there must be a Condorcet winner if the set is not a cycle.

Combining the two conditions, we get that if the set of possible outcomes is finite, then either a Condorcet winner exists or there is a cycle since there are only two possible outcomes for the set because a set is either a cycle or not a cycle.

Writers 0 through 12 vote for 7, and 13 through 21 vote for 18, so 7 bobas wins.

Suppose we have proposals for i and j bobas where $j > i$.

. If $ji2$, then every writer from 0 to $\lfloor \frac{j-i}{2} \rfloor + i$ will vote for i and every writer from $\lfloor \frac{j-i}{2} \rfloor + i + 1$ to 21 will vote for j. Thus, $\lfloor \frac{j-i}{2} \rfloor + i + 1$ people vote for i and $21 - \frac{j-i}{2} - i$ people will for j. If $i = 10$, then we have

$$11 + \frac{j-i}{2} \geq 11 - \frac{j-i}{2}.$$

If $j = 10$, then we have that the number of people that vote for i is

$$\frac{10-i-1}{2} + i + 1$$

and the number of people that vote for 10 is

$$21 - \frac{10-i-1}{2} - i.$$

We have that

$$21 - \frac{10-i-1}{2} - i > \frac{10-i-1}{2} + i + 1 \implies 11 > i,$$

which is true. Thus, 10 always gets the most votes. The same argument can be repeated when i and j have the same parity without the floors.

Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be the ideal preference of person n where n is odd. Let $m = \frac{n+1}{2}$. We need to show that any option that a_m likes mote will always get the most votes.

. Suppose that two numbers x and y are proposed such that a_m prefers x over y. We split into cases.

. Case 1: $a_m \leq x < y$. In this case, a_i for $i \in 1, 2, \ldots, \frac{n+1}{2}$ will be guaranteed to pick x. Thus at least $\frac{n+1}{2}$ people will always be for x, which is a majority.

. Case 2: $y < x \leq a_m$. Essentially the same as above.

. Case 3: $x \leq a_m < y$. Note that $a_m - y \leq a_{m-1} - y$, while a_{m-1} is either closer to x or further left of it, meaning a_{m-1} will always pick x. Similarly, a_i will pick x for $i \in 1, 2, \ldots, \frac{n+1}{2}$, which means again x will win.

. Case 4: $y < a_m \leq x$ and $a_m - x < a_m - y$. Essentially the same as above.

Suppose we have problem writers with ideal boba numbers of 1,2,3,4,5,6,7 and three outcomes 3,4,5. Here the problem writers from 1 through 3 will vote for 3, only 4 will vote for 4, and 5 through 7 will vote for 5. Clearly 4 gets the least number of votes here despite being the median problem writer value.

if Isaack suggests 19, problem writers 1-8 will vote for 7, 9-14 will vote for 10, and 15-21 will vote for 19, meaning there are 8,6, and 7 votes for Erick's, Vicktor's, and Isaack's proposals, respectively.

There is no such proposal that Isaack can make to get Erick to win. Suppose Isaack proposes a number i where $15 <= i <= 21$. Then Erick's proposal of 15 will get the votes from 15 through $i/2 + 7$, while Isaack will get the rest of the votes. Here Erick cannot possibly win since he needs a minimum of 8 votes, and he can get $(21/2 + 7) - 15 + 1 = 3$ votes max in this scenario. If Isaack proposes a number $i < 14$ on the other hand, Erick is guaranteed all of the votes from 15 through 21 which is 7 but can't gather any more since Vicktor and Isaack will take everything $<= 14$. Then Vicktor gets, for some positive integer i where 1 ¡= i ¡= 7, i votes and Isaack gets $14 - i$ votes but for Erick to win, $7 > i, 7 > 14 - i$ implies $7 > i > 7$, a contradiction.

There is no such proposal that Isaack can make to get Erick to win. Suppose Isaack proposes a number i where $15 <= i <= 21$. Then Erick's proposal of 15 will get the votes from 15 through $i/2 + 7$, while Isaack will get the rest of the votes. Here Erick cannot possibly win since he needs a minimum of 8 votes, and he can get $(21/2 + 7) - 15 + 1 = 3$ votes max in this scenario. If Isaack proposes a number $i < 14$ on the other hand, Erick is guaranteed all of the votes from 15 through 21 which is 7 but can't gather any more since Vicktor and Isaack will take everything $<= 14$. Then Vicktor gets, for some positive integer i where 1 ¡= i ¡= 7, i votes and Isaack gets $14 - i$ votes but for Erick to win, $7 > i, 7 > 14 - i$ implies $7 > i > 7$, a contradiction.

Suppose we have $a_1, a_2, \ldots, a_n$ as outcomes. If the choice function is efficient, then if $a_i \succ_1 a_j$, we have $a_i \succ a_j$, which is clearly a dictatorship. Similarly, if we have $a_i \succ a_j$, then we must have $a_i \succ_1 a_j$, otherwise the social outcome would be different from the only individual's preference.

Consider the example of a dictatorship with 1 as the dictator, then it follows that $x \succ y$ and $x \succ_1 y$ and $y \succ_2 x$. This can be denoted with the fact that if there are two political coandidates and there is a strong pereference towards one that they must hold a literal dictatorship.

Since 1 is a dictator, their interests align with the majority, and thus since society prefers $x \succ y$ for some events x and y, it implies that for some other relatiosn a and b the preference of 1 would be aligning with the rest of society, and thus $a \succ_1 b$ implies $a \succ b$.

Let the individuals be 1 and 2, then we have that although completeness allows $x \succ_1 y$ and $y \succ_2 x$, society prefers $x \succ y$, and since there are only two individuals 1's preference corresponds to that of society, so 1 is a dictator.

8. Conclusion

Voting is certainly a commonplace event in the modern world. It is nice to theorize among ideal scenarios in order to gain a deeper understanding of the mechanism behind it. Impossibility is one nice consequences; the other is paradoxes, which are similar but differ in significant ways, more information which can be found in my paper.

References

[1] Haris Aziz and Alexander Lam. Obvious manipulability of voting rules. In *Algorithmic Decision Theory: 7th International Conference, ADT 2021, Toulouse, France, November 3–5, 2021, Proceedings 7*, pages 179–193. Springer, 2021.

[2] Salvador Barberà, Jordi Massó, and Alejandro Neme. Voting under constraints. *journal of economic theory*, 76(2):298–321, 1997.

[3] Donald E Campbell. Computational criteria for voting systems. *British Journal of Political Science*, 7(1):85–98, 1977.

[4] Peter Gärdenfors. A representation theorem for voting with logical consequences. *Economics & Philosophy*, 22(2):181–190, 2006.

[5] John Geanakoplos. Three brief proofs of arrow's impossibility theorem. *Economic Theory*, 26(1):211–215, 2005.

[6] Wesley H Holliday, Chase Norman, Eric Pacuit, and Saam Zahedian. Impossibility theorems involving weakenings of expansion consistency and resoluteness in voting. *Mathematical Analyses of Decisions, Voting and Games*, 795:87, 2024.

[7] Eitan Muller and Mark Satterthwaite. An impossibility theorem for voting with a different interpretation. Technical report, Discussion Paper, 1976.

[8] Elizabeth Maggie Penn. Impossibility theorems and voting paradoxes in collective choice theory. *Wiley Encyclopedia of Operations Research and Management Science*, 2011.

[9] Alan D Taylor. The manipulability of voting systems. *The American Mathematical Monthly*, 109(4):321–337, 2002.

[10] Sergei Vasiljev. Cardinal voting: the way to escape the social choice impossibility. *Vasiljev SA CARDINAL VOTING: THE WAY TO ESCAPE THE SOCIAL CHOICE IMPOSSIBILITY. Young Scientist USA. Social science. Auburn, USA*, pages 80–85, 2014.

[1] [2] [3] [4] [5] [6] [7] [8] [9] [10]

!

!

!

!

!

!

!

!

!