Outer Automorphism of S_6

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What is a symmetric group?

Definition 0.1

- The **symmetric group** is defined as the elements of the group that are permutations on the given set (i.e., bijective maps from the set to itself).
- The **product** of two elements is their composite as permutations, i.e., function composition.
- The **identity** element of the group is the identity function from the set to itself.
- The **inverse** of an element in the group is its inverse as a function.

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Example

Example: $S_3 = \{(), (12), (13), (23), (123), (132)\}$

Automorphism group

Definition 0.2

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Example

$Aut(\mathbf{S}_2)$ where $\mathbf{S}_2$ is $\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

- $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ is the identity
- An automorphism f must send $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ to itself
- f must send $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ to itself
- f is the identity e which means that $Aut(\mathbf{S}_2)$ is trivial

Automorphism group

Example

Let ϕ: $\mathbb{Z}_8 \to \mathbb{Z}_8$ and $\phi(x) = 3x \pmod 8$. This is bijective since we can clearly show that each element maps to another element.

Original	Mapped
0	0
1	3
2	6
3	1
4	4
5	7
6	2
7	5

Homomorphic: $\phi(x + y) = 3(x + y) = 3x + 3y = \phi(x) + \phi(y)$.

Inner Automorphism

Definition 0.3

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Example 1

Let $f(x)$ be defined as $(12)x(12)$. Then f(123) = (12)(123)(12) = (132). $f(x)$ is an inner automorphism of S_3

Complete Groups

Definition 0.4

$Z(G) = \{z \in G \mid \forall g \in G, zg = gz\}$ where $Z(G)$ is the **center** of the group

Example

The quartenion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ has $Z(Q_8) = \{\pm 1\}$ which means that $1x = x1$ and $-1x = x(-1)$ is true for all $x \in Q_8$.

Complete Groups

Definition 0.5

A group is said to be **centerless** if $Z(G)$ is trivial; i.e., consists only of the identity element.

Definition 0.6

A group G is **complete** if it is centerless and every automorphism of G is inner.

Complete Groups

Theorem 0.7

S_n is complete for $n \neq 2, 6$

Remark 0.8

Permutations with the same cycle length when decomposed into transpositions belong to the same conjugacy class

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Permutations with the same cycle length when decomposed into transpositions belong to the same conjugacy class

- Let T_k be the conjugacy class in S_n consisting of products of k disjoint transpositions.
- A permutation π is an involution if and only if it lies in some T_k.
  - If $f \in \mathrm{Aut}(S_n)$, then $f(T_1) = T_k$ for some k.
- It suffices to show $|T_k| \neq |T_1|$ for $k \neq 1$.
  - This is true for $n \neq 6$.
  - For $n = 6$, it turns out that $|T_1| = |T_3|$ is the only exception.

Transitive Group Actions

Definition 0.9

A group G **acts** transitively on a set X if for any $x, y \in X$ there is some $g \in G$ such that $g \circ x = y$

- **Example 1:** S_n acts on $\{1, 2, 3, \ldots, n\}$. For each $i, j \in \{1, 2, ..., n\}$, there is a $\tau \in G$ with $\tau(i) = j$

Outer Automorphism

Definition 0.10

Outer automorphism is any automorphism that is not inner

Pentagons

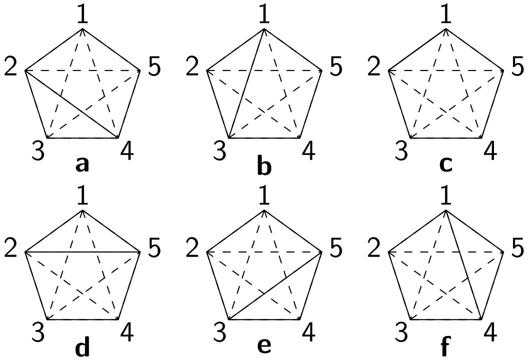

Figure 1: Pentagons with various edges and diagonals

Visualization

Mystic Pentagons

- 12 ways to two-color edges of a complete graph on 5 vertices
- Each element of S_5 induces a permutation of six mystic pentagon pairs via action on vertices: $i : S_5 = S_{\{1,\ldots,5\}} \to S_{\{a,\ldots,f\}} = S_6$.
- Inclusion: every element in $S_{\{1,\ldots,5\}}$ is in $S_{\{a,\ldots,f\}}$.
- Homomorphism between $S_{\{1,\ldots,5\}}$ and $S_{\{a,\ldots,f\}}$.
- Not usual inclusion: (12) induces permutation $(ad)(bc)(ef)$ (not a transposition).
- $S_6 = S_{\{a,\ldots,f\}}$ acts on six cosets of $i(S_5)$, inducing $f : S_{\{a,\ldots,f\}} \to S_{\{1,\ldots,6\}}$.
- Outer automorphism

Sylow's theorem

Definition 0.11

If p^k is the highest power of a prime p dividing the order(number of elements) of a finite group G, then a subgroup of G of order p^k is called a sylow $\mathbf{p-subgroup}$ of G.

Theorem 0.12

For a group G with order divisible by p, there exists a p-Sylow subgroup. Let x be the number of p-Sylow subgroups. Then,

$$x \equiv 1 \pmod p \text{ and } x \mid |G|.$$

All p-Sylow subgroups are conjugate. (For every pair of p-Sylow subgroups H and K, there exists $g \in G$ with $g^{-1}Hg = K$.)

Construction of $Out(S_6)$

- The 5-Sylow subgroups of S_5 are exactly the subgroups generated by a 5-cycle. Let X be the set of these subgroups.
- Then, $|X| \equiv 1 \pmod 5$ and $|X| \mid 120$ so $|X| = 6$.
- Consider the action of S_5 on X by conjugation (g sends X to $g^{-1}Xg$).
  - By Sylow's theorem, this action is transitive.
  - The action gives a homomorphism f: $S_5 \to S_6$, since $|X| = 6$.
  - $ker(f)$ (elements that f maps to identity) forms a normal subgroup, so $ker(f) = \{A_5, S_5, e\}$
  - Since the action is transitive, $ker(f) \leq |S_5|/6 = 20$.
  - Hence $|ker(f)| =$ e.
  - Thus, im(f) is a transitive 120-element subgroup of S_6.

Conclusion

I would like to thank Emma Cardwell and Simon Rubinstein. Thank you for listening.