Introduction to Martingales

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Outline

Introduction

- In this presentation, I'll talk about **Martingales**, which is a probability theory based on a fair game (we'll cover that on the next slide).
- Introduced by Paul Lévy in 1934 and later by Ville in 1939.

Fair Game

Let's consider a bettor participating in a game involving the flipping of a fair coin. In this game, the bettor earns \$1 for a heads outcome and loses \$1 for a tails outcome. This means that if the better flips the coin once and it lands on tails, they would lose a dollar. Conversely, if it lands on heads, they would gain a dollar.

Quick Definition of Martingales

- In simple terms, a martingale is a process where the conditional expectation of the next value is equal to the present value.
- Definition: X is a martingale if

$$X_n = \mathbb{E}[X_{n+1} | \mathcal{F}_n]$$

Outline

Required variables and their definition

- Sample Space $= \Omega$
- Sigma Algebra $= \mathcal{F}$
- Indicator Function $= \mathbb{1}$
- Expectation $= \mathbb{E}$
- Probability Measure $= \mathbb{P}$

Sigma-Algebras

A system $A \subseteq P(X)$, where A is a collection of elements of subsets of X. This collection is called σ-algebra. A σ-algebra is a collection of subsets of the sample space that includes the empty set and the sample space itself.

- $\varnothing, X \subseteq P(X)$
- If $B \in A$ then $B^C = X/B \in A$.
- $B, D \in A \to B \cup D \in A$, $B \cap D \in A$

Random Variables

A random variable is a measurable function from the sample space to the real numbers.

Sample Space

A probability measure ($\mathbb{P}$) on the sample space satisfies $\mathbb{P}(\Omega) = 1$. Therefore, the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

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Indicator Functions

Let A be a subset of Ω. The indicator function of A is the function $\mathbf{1}_A : \Omega \to \mathbb{R}$ defined as:

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \notin A \end{cases}.$$

Indicator functions are way to analyze and construct new martingales.

Simple Functions

Simple functions are measurable functions that take on a finite number of values. They can be written as:

$$g = \sum_{i=1}^{n} a_i \mathcal{X} A_i$$

Steps of Integrating

$$g = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$$

$$\int g d\mu = \int \sum_{i=1}^{n} a_i \mathcal{X}_{A_i} d\mu$$

$$\int \sum_{i=1}^{n} a_i \mathcal{X}_{A_i} d\mu = \sum_{i=1}^{n} a_i \int \mathcal{X}_{A_i} d\mu$$

$$\int \mathcal{X}_{A_i} d\mu = \mu\left(A_i\right)$$

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Formula of the Lebesgue Integration

$$\int \mathrm{gd}\mu = \sum_{\mathrm{i=1}}^{\mathrm{n}} a_i \mu(Ai), \text{ where } \mu \text{ is a measure on } \Omega$$

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Outline

What is expectation? Conditional Expectation?

- Conditional expectation, denoted $\mathbb{E}[Y|X]$, represents the expected value of a random variable Y given the knowledge of another variable X.
- It is used to update our expectation of Y based on the additional information provided by X.

Kolmogorov's Theorem on Conditional Expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X is an integrable random variable; $\mathbb{E}|X| < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub sigma-algebra. Take random variable Y such that it follows these properties:

- Y is $\mathcal{G}$-measurable.
- $\mathbb{E}|Y| < \infty$
- $\forall A \in \mathcal{G}$
- $\int_A X dP = \int_A Y dP = \int_A \mathbb{E}(X|\mathcal{G}) dP$

The random variable Y is denoted by $\mathbb{E}(X|\mathcal{G})$ and is called the conditional expectation of X given $\mathcal{G}$.

Tower Rule

$(\Omega, \mathcal{F}, \mathbb{P})$; $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$ (These are all σ-algebras). Both $\mathcal{G}$ and $\mathcal{H}$ are sub-sigma-algebras. Let X be a random variable, $\mathbb{E}[X] < \infty$.
Then the Tower rule states:

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}[X|\mathcal{G}]$$

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Martingale Property

We consider a sequence of random variables $X = \{X_n; n \geq 0\}$ such that

- $\{X_n; n \geq 0\}$ is adapted.
- $X_n \in L^1(\Omega)$ for all $n \geq 0$.

Then

- X is a martingale if $X_n = \mathrm{E}\,[X_{n+1} \mid \mathcal{F}_n]$.

Proof

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$$\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = \mathbb{E}\left[X_n + Y_{n+1} \mid \mathcal{F}_n\right] = X_n + \mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] = X_n + 0 = X_n$$

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This confirms that the process $\{X_n\}_{n\geq 0}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$

Optional Stopping

Doob's Optional Stopping: Let T be a stopping time, X is a martingale. If it follows these conditions:

- T is bounded OR X is bounded and T is infinite almost surely OR $\mathbb{E}[T] < \infty$ and $|X_n - X_{n-1}| \leq k$

Then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$

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Applications in the Stock Market

This martingale theory can be implemented into the stock market. For instance, the Martingale Strategy says to double your initial bet if you are down. In the sense of the stock market, if the one stock you bought at 10 dollars, is now 9 dollars. You buy another stock at the price. However, if it drops again to 8 dollars, you buy another two stocks. Finally, when the stock goes up to 10 dollars, not only did you make you initial stock back, but made profit from the other three stocks you bought at a lower price.