Borsuk-Ulam Theorem

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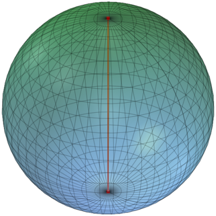
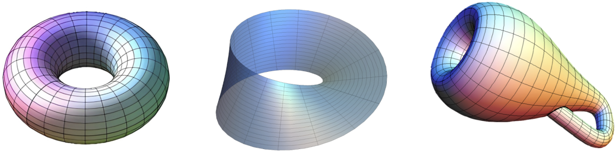
What is the Borsuk-Ulam theorem?



The Borsuk–Ulam theorem states that for every continuous mapping $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(x)$.

Topological Space

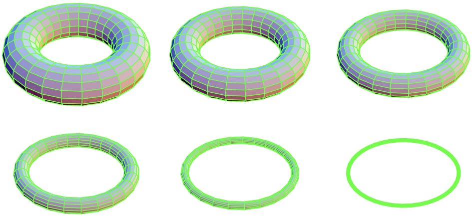


A topological space, also called an abstract topological space, is a set X together with a collection of open subsets T that satisfies the four conditions:

- The empty set ϕ is in T.
- X is in T
- The intersection of a finite number of sets in T is also in T.
- The union of an arbitrary number of sets in T is also in T.

Homotopy

Two mathematical objects are considered homotopic if one can transform continuously into the other.



Definition

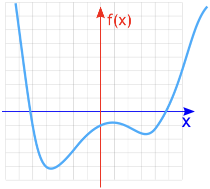
A homotopy between two continuous functions $f, g : X \to Y$ from one topological space X to another topological space Y is defined as a continuous map $H : X \times [0, 1] \to Y$ such that:

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, 1) = g(x)$$

for all $x \in X$. Intuitively, H describes a continuous transformation from f to g over the time interval $[0, 1]$

Continuous Function

A continuous function is a function such that a small variation of the argument induces a small variation of the value of the function. This implies there are no abrupt changes in value, known as discontinuities.



Definition

For a function $f(x)$ to be continuous at a point a it must satisfy the following conditions:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$
4. For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$, $x \neq a$ implies that $|f(x) - f(a)| < \varepsilon$

Proving Borsuk-Ulam

Proof.

If there were to exist a function f, and a point on the sphere $\overrightarrow{\text{p}}$, $f(\overrightarrow{\text{p}}) = f(\overrightarrow{\text{-p}})$ as long as the function is continuous.Fundamentally the equation can be rearranged in the following manner:

$$f(\overrightarrow{\text{p}}) - f(-\overrightarrow{\text{p}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can now utilize a new function g, which is equivalent to the left hand side of the equation displayed above:

$$g(\overrightarrow{\text{p}}) := f(\overrightarrow{\text{p}}) - f(-\overrightarrow{\text{p}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

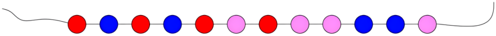
Now, we can observe a characteristic of the function g, which is:

$$-g(\overrightarrow{\text{p}}) = g(-\overrightarrow{\text{p}})$$

$$g(-\overrightarrow{\text{p}}) = f(-\overrightarrow{\text{p}}) - f(\overrightarrow{\text{p}}) = -g(\overrightarrow{\text{p}})$$

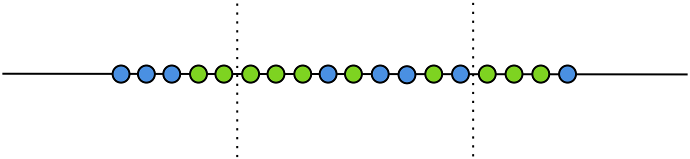
Therefore, there is a point $\overrightarrow{\text{p}}$ where $g(\overrightarrow{\text{p}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ □

What is the Necklace Problem?



In this problem, two thieves have stolen a valuable necklace consisting of several different types of jewels. There are an even number of each type of jewel and the thieves wish to split each jewel type evenly amongst the two of them. The catch is that they must do so by splitting the necklace into some number of contiguous segments and distribute the segments between the two of them.

Using Borsuk-Ulam



Thank you!