BORSUK-ULAM THEOREM

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Abstract

This expository paper introduces the principles orchestrated by the Borsuk-Ulam theorem, a concept in algebraic topology, introduced by mathematicians Karol Borsuk and Stanisław Ulam. Through this paper, I will explore the claims of the theorem, the applications it poses, the proofs, and its further implications.

Acknowledgements

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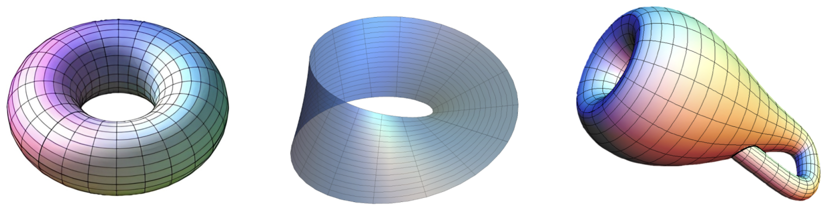
I Background

I.I Topological Spaces

To fully comprehend the Borsuk-Ulam theorem, we must review the founding concepts upon which the theorem is built. We will start with defining a topological space, a mathematical structure which displays the properties of continuity and closeness within a given set of points.



Definition 1. A topological space is a pair $(X, \mathcal{O})$ where X is a ground set and $\mathcal{O} \subseteq 2^X$

A topological space, also called an abstract topological space, is a set X together with a collection of open subsets T that satisfies the four conditions:

(1) The empty set $\oslash$ is in T.
(2) X is in T.
(3) The intersection of a finite number of sets in T is also in T.
(4) The union of an arbitrary number of sets in T is also in T.

Alternatively, T may be defined to be the closed sets rather than the open sets, in which case conditions 3 and 4 become:

(3) The intersection of an arbitrary number of sets in T is also in T.
(4) The union of a finite number of sets in T is also in T.

These axioms are designed so that the traditional definitions of open and closed intervals of the real line continue to be true.

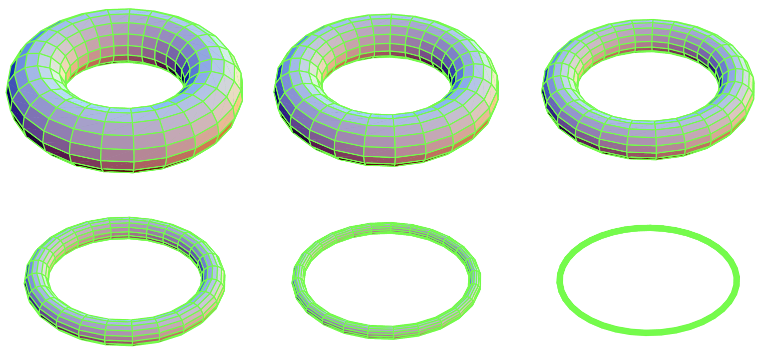
I.II Homotopy

Two mathematical objects are homotopic if they can be continuously transformed into each other. For instance, the real line can be smoothly deformed into a single point. Conversely, although the circle cannot be shrunk down to a point, it is homotopic to a solid torus. The core idea of homotopy revolves around the continuous mappings between these objects.

Definition 2. In mathematics, specifically in topology, a homotopy between two continuous functions $f, g : X \to Y$ from one topological space X to another topological space Y is defined as a continuous map $H : X \times [0,1] \to Y$ such that:

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x)$$

for all $x \in X$. Intuitively, H describes a continuous transformation from f to g over the time interval $[0,1]$.



Continuous functions f and g are homotopic if there exists a homotopy H that continuously transforms f into g, as explained earlier. Homotopy defines an equivalence relation on the set of all continuous functions from X to Y. This relation maintains compatibility with function composition: if $f_1, g_1 : X \to Y$ are homotopic and $f_2, g_2 : Y \to Z$ are also homotopic, then their compositions $f_2 \circ f_1$ and $g_2 \circ g_1 : X \to Z$ are likewise homotopic.

I.III Homeomorphism

In mathematics, a homeomorphism refers to a correspondence between two geometric figures, surfaces, or other objects that is established by a continuous, bijective mapping with a continuous inverse. In the figure provided, the vertical projection creates a one-to-one correspondence between the straight segment x and the curved interval y. If x and y are topologically equivalent, there exists a function $h : x \to y$ that is continuous, surjective (each point of y corresponds to a point of x), injective (each point of x corresponds to a unique point of y), and its inverse h^{-1} is also continuous. Such a function h is termed a homeomorphism.

A topological property is one that remains unchanged under a homeomorphism. Examples include connectedness, compactness, and, for a domain in the plane, the number of components of its boundary. Topological spaces provide the most general context where homeomorphisms can be defined. Two spaces are considered topologically equivalent if there exists a homeomorphism between them.

Definition 3. A homeomorphism between two topological spaces X and Y is a bijective function $h : X \to Y$ such that both h and its inverse $h^{-1} : Y \to X$ are continuous mappings.

Conditions

(1) h is bijective (one-to-one and onto).
(2) h is continuous: For every open set U in X, the set $h(U)$ is open in Y.
(3) h^{-1} is continuous: For every open set V in Y, the set $h^{-1}(V)$ is open in X.

Essentially, a homeomorphism preserves the topological structure of the spaces it connects, meaning that it transforms open sets in X into open sets in Y and vice versa, while maintaining the one-to-one correspondence between points and the continuity of mappings.

I.IV Continuous Functions

In mathematics, a continuous function is defined such that small changes in the input cause small changes in the output. This property ensures there are no sudden jumps or discontinuities in the function's values. To be precise, a function is continuous if for any small change in its output, there exists a small enough change in its input that guarantees a correspondingly small change in the output. Conversely, a function that does not satisfy this condition is called discontinuous.

Conditions

For a function $f(x)$ to be continuous at a point a it must satisfy the following conditions:

(1) $f(a)$ exists
(2) $\lim_{x \to a} f(x)$ exists
(3) $\lim_{x \to a} f(x) = f(a)$
(4) For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$, $x \neq a$ implies that $|f(x) - f(a)| < \varepsilon$

II Borsuk-Ulam Theorem

The Borsuk-Ulam theorem is a theorem which states the following:

Theorem 1. For every continuous map $f : S^n \to R^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.

Essentially, for any continuous mapping from an n-dimensional sphere to n-dimensional Euclidean space, there exists at least one pair of antipodal points mapped to the same point.

If there were to exist a function f, and a point on the sphere $\overrightarrow{\mathrm{p}}$, $f(\overrightarrow{\mathrm{p}}) = f(\text{-}\overrightarrow{\mathrm{p}})$ as long as the function is continuous. Here, -$\overrightarrow{\mathrm{p}}$ is the antipodal point of $\overrightarrow{\mathrm{p}}$, located directly on the opposite side of the sphere. Fundamentally, the equation can be rearranged in the following manner:

$$f(\overrightarrow{\mathrm{p}}) - f(-\overrightarrow{\mathrm{p}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can now utilize a new function g, which is equivalent to the left hand side of the equation displayed above:

$$g(\overrightarrow{\mathrm{p}}) := f(\overrightarrow{\mathrm{p}}) - f(-\overrightarrow{\mathrm{p}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Essentially the function g, maps some point on the sphere $\overrightarrow{\mathrm{p}}$, onto the origin in the two-dimensional space. Now, we can observe a characteristic of the function g, which is:

$$-g(\overrightarrow{\mathrm{p}}) = g(-\overrightarrow{\mathrm{p}})$$
$$g(-\overrightarrow{\mathrm{p}}) = f(-\overrightarrow{\mathrm{p}}) - f(\overrightarrow{\mathrm{p}}) = -g(\overrightarrow{\mathrm{p}})$$

To put this in words, on the antipodal point on the sphere, the value of g in the output space is reflected through the origin of the output space. We can visualize this as rotating the output of g 180 degrees about the origin. Now, let us picture the equator of the earth, and take $\overrightarrow{\mathrm{p}}$ to be a point on the equator. If we begin travelling along the equator, starting from the point $\overrightarrow{\mathrm{p}}$, once we have covered half the distance, we have reached the antipodal point, $-\overrightarrow{\mathrm{p}}$. In the output space, you have reached the reflection of the starting point through the origin. Now, as we continue travelling along the remaining half of the equator, the second half of the output path must be the reflection of the first half. As discussed before, this can also be interpreted as the 180 degree rotation of the first half of the output path. The objective we have, is for one of the points on this continuous function in the output space, to pass through the origin. If this is not the case with the function, what we have is a path which travels around the origin at least once. Now, we can once again visualize the equator on the input space, and continuously deform it, until it is a narrow orbit about the north pole. This deformation is reflected in the output space, which now exhibits the function continuously deforming to a single point. As the continuous function in the output space is centred around the origin, during the deformation, it must cross the origin. Therefore, there is a point $\overrightarrow{\mathrm{p}}$ where $g(\overrightarrow{\mathrm{p}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Theorem 1.1 (Borsuk's Original Formulation)

For every $n \geq 0$, the following statements are equivalent, and true:

(BU1a) (Borsuk [Bor33, Satz II]3) For every continuous mapping $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(x)$.
(BU1b) For every antipodal mapping $f : S^n \to \mathbb{R}^n$ (that is, f is continuous and $f(x) = f(x)$ for all $x \in S^n$) there exists a point $x \in S^n$ satisfying f(x) = 0.
(BU2a) There is no antipodal mapping $f : S_n \to S_{n-1}$.
(BU2b) There is no continuous mapping $f : B_n \to S_{n-1}$ that is antipodal on the boundary, i.e., satisfies $f(x) = f(x)$ for all $x \in S^{n-1} = \epsilon B^n$.
(LS-c) (Lyusternik and Shnirel'man [LS30], Borsuk [Bor33, Satz III]) For any cover $F_1, ..., F_{n+1}$ of the sphere S^n by $n + 1$ closed sets, there is at least one set containing a pair of antipodal points (that is, $F_i \cap (F_i) \neq \phi$)
(LS-o) For any cover $U_1, ..., U_{n+1}$ of the sphere S_n by $n + 1$ open sets, there is at least one set containing a pair of antipodal points.

II.I Lusternik-Schnirelmann

We can also understand the Borsuk-Ulam theorem through the Lusternik-Schnirelmann theorem.

Theorem 2. Suppose S^n is covered by $n + 1$ closed sets R_1,...,R_{n+1}. Then one of the sets contains a pair of antipodal points.

Proof. Define $d_i(x)$ as minimum distance from x to R_i

(1) Let $f : S^n \to \mathbb{R}^n$ be defined as $f(x) = (d_1(x), ..., d_n(x))$
(2) By Borsuk-Ulam, exists an $x \in S^n$ such that $d_i(x) = d_i(-x)$ for $1 \leq i \leq n$

Two Cases:

(1) $d_i(x) = d_i(-x) = 0$ for some i. Then both x and $-x$ are in R_i
(2) $d_i(x) = d_i(-x) > 0$ $\forall i$ for $1 \leq i \leq n$, hence both x and $-x$ are in R_{n+1}

II.II Brouwer's Fixed Point

The Brouwer's Fixed Point theorem is a theorem which claims that any given continuous function f propping a compact convex set onto itself includes no less than one fixed point (a point x where $f(x) = x$). We can visualize this theorem in a host of different ways. For example, imagine you have two identical maps of the room that you are currently in. If you crumple one of the maps and place it on top of the other, there will be at least one point on the crumpled version which aligns with the location on the map below. Alternatively, you can imagine you have two maps of a country, of different sizes, and place them on top of another, the same phenomenon will be observed. This will occur with any sort of mapping, even when there is a difference in rotation.

To grasp this theorem, we must understand compact and convex sets.

II.II.I Compact Sets

A compact set has two properties, it is bounded and closed:

Properties

(1) All points are within a constant distance of each other. There is a number N such that $f(x) \leq N$ regardless of the value of x.
(2) The sets includes all of its limit points. For example if a sequence of points approaches a value N, the set contains the value N.

II.II.II Convex Sets

A set is convex if for any combination of two points in the set their convex combination is included in the set. Some elementary examples of a convex set include a closed disk, any convex region in the plane, and a closed interval in real numbers.

Definition 4 A set C is convex if the line segment between 2 points in C lies in C, i.e $\forall x_1, x_2 \in C, \forall \theta \in [0, 1]$

$$\theta x_1 + (1 - \theta)x_2 \in C$$

This can be generalized from 2 points to any number of points. A convex combination of points, $x_1, x_2, x_3, x_4, ..., x_n \in C$ is a point in the form $\theta_1 x_1 + \theta_2 x_2 + ... + \theta_k x_n$, where $\theta_i \geq 0$, $i = 1...n, \sum_{i=1}^{n} \theta_i = 1$

To further this, C is convex iff $\forall x_i \in C, \theta_i \geq 0, i = 1, 2, ..., \sum_{i=1}^{\infty} \theta_i = 1$:

$$\sum_{i=1}^{\infty} \theta_i x_i \in c$$

if there is a converging series.

C is convex iff for X over C, $\mathbb{P}(X \in C) = 1$, its expectation is also in C:

$$\mathbb{E}(X) \in C$$

The fixed point theorem states that if R is a compact convex set, and $f : R \to R$ is continuous, then x exists such that $f(x) = x$. Both of the aforementioned conditions are compulsory. For example, let us take the given function:

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x + 1$$

This function does map $\mathbb{R}$ to itself, but however, it does not contain a fixed point. Similarily we can look at the following equation:

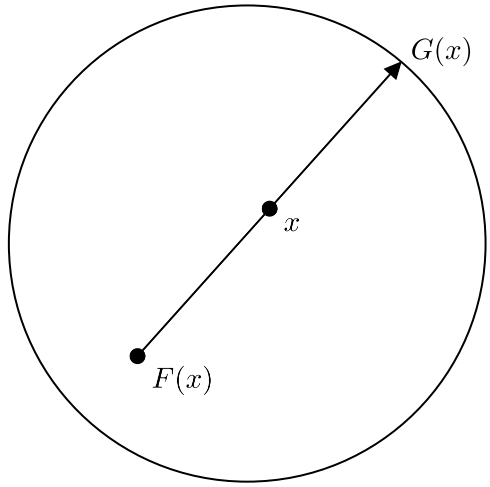
$$f : (0, 1) \to (0, 1), f(x) = \frac{x + 1}{2}$$

This equation also maps the region to itself, but has no fixed points.

II.II.III Proof

Theorem 3. Let $f : D^2 \to D^2$ be a continuous map, where $D^2 = (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1$ is a two-dimensional disc. Then, there is a point $x \in D^2$ such that $F(x) = x$.

Proof. Assume, for contradiction, $F(x) \neq x$ for all $x \in D^2$. We can now define $G : D^2 \to \partial D^2$ (where ∂D^2 is the boundary circle) like so: Take the line beginning at $F(x)$, going through x and let $G(x)$ to be the point of intersection of this line with ∂D^2.



We make the following declarations:

Claims

(1) The function G is continuous
(2) $G(x) = x$ if $x \in \partial D^2$, regardless of the position of $F(x)$, the line from $F(x)$ to x intersects the boundary at x, so $G(x) = x$.

From the statements above, we arrive at a contradiction. Let $i : \partial D^2 \to D^2$ be the inclusion map of the boundary. $G(i)$ is equivalent to the identity map on ∂D^2 by (2). The following is implied:

$$id = G_* \circ i_* : \pi_1(\partial D^2) \to \pi_1(\partial D^2)$$

We now have $\pi_1(\partial D^2) = \mathbf{Z}$, which shows that the identity map on the integers factors through the map $i_* : \pi_1(\partial D^2) \to \pi_1(D^2) = 0$:

$$\begin{array}{ccccc} \pi_1(D2) & \xrightarrow{i_*} & \pi_1 D^2 & \xrightarrow{G_*} & \pi_1(D2) \\ \| & & \| & & \| \\ \mathbf{Z} & & 0 & & \mathbf{Z} \end{array}$$

The above relation implies that $G_* \circ i_* = 0$. This is not possible, as $n = G_*(i_*(n)) = G_*(0) = 0$ for all $n \in \mathbf{Z}$ is implied. Henceforth, we arrive at a contradiction. Brouwer's Fixed Point theorem has been proved modulo (1).

Now, we have to show that G is a continuous function. G can be written as $H \circ j$, like so:

(1) $j : D^2 \to D^2 \times D^2 \setminus \{(x, x) : x \in D^2\}$ is the map $j(x) = (x, F(x))$.

(2) $H : D^2 \times D^2 \setminus \{(x,x) : x \in D^2\}$ is the map defined as follows. If we take 2 distinct points $x \neq y$ in D^2, let $H(x,y)$ mark the point where the line from y through x intersects ∂D^2.

By definition, we have $G(x) = H(x, F(x)) = H(j(x))$. Using product topology, j is continuous, and therein H is continuous.

H can be written in coordinate form. The line starting from y that goes through x can be written as $y + t(x-y)$. The condition that the line meets the boundary of the disc is as follows:

$$\begin{aligned} |y + t(x-y)|^2 &= 1 \\ t^2|x-y|^2 + 2t(x-y)\cdot y + |y|^2 - 1 &= 0 \\ t_{\pm} = \frac{-2(x-y)\cdot y \pm \sqrt{4((x-y)\cdot y)^2 - 4|x-y|^2(|y|^2-1)}}{2|x-y|^2} \end{aligned}$$

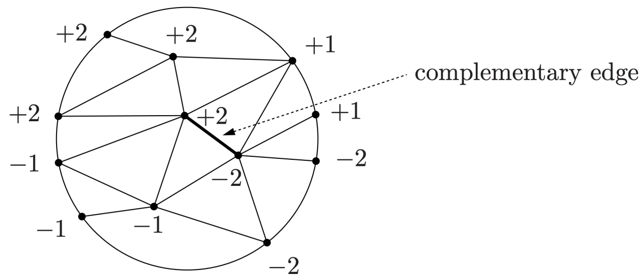
Therefore, the point H, is now written as rational functions and surds, all of which are continuous.

II.III Tucker's Lemma

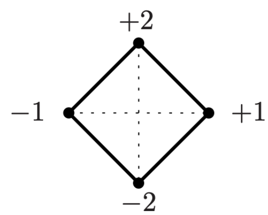
Here we deduce the Borsuk–Ulam theorem from a combinatorial statement known as Tucker's lemma. This lemma concerns the labeling of vertices in triangulations of the n-dimensional ball. Interestingly, Tucker's lemma can also be inferred from the Borsuk–Ulam theorem.

Let T denote a (finite) triangulation of the n-dimensional ball B^n. We say T is antipodally symmetric on the boundary if the set of simplices of T contained in $S^{n-1} = \partial B^n$ forms an antipodally symmetric triangulation of S^{n-1}; that is, if $\sigma \subset S^{n-1}$ is a simplex of T, then $-\sigma$ is also a simplex of T.

Theorem 4 Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Suppose $\lambda : V(T) \to \{+1, -1, +2, -2, \ldots, +n, -n\}$ is a vertex labeling of T satisfying $\lambda(-v) = -\lambda(v)$ for every vertex $v \in \partial B^n$ (i.e., λ is antipodal on the boundary). Then there exists an edge in T (a 1-simplex) whose two vertices are labeled with opposite signs.



We reformulate Tucker's lemma using simplicial maps into the boundary of the crosspolytope. Let $\diamondsuit^{n-1}$ denote the (abstract) simplicial complex with vertex set $V(\diamondsuit^{n-1}) = \{+1, -1, +2, -2, \ldots, +n, -n\}$, where a subset $F \subseteq V(\diamondsuit^{n-1})$ forms a simplex whenever there is no $i \in [n]$ such that both $i \in F$ and $-i \in F$. One can recognize $\diamondsuit^{n-1}$ as the boundary complex of the n-dimensional crosspolytope. This notation is particularly suggestive in the case $n = 2$:



Theorem 5 Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Then there does not exist a map $\lambda : V(T) \to V(\diamondsuit^{n-1})$ that is a simplicial map from T into $\diamondsuit^{n-1}$ and is antipodal on the boundary.

A map $B^n \to S^{n-1}$ that is antipodal on the boundary, does not exist. Deriving Tucker's lemma, in the form above, is immediate: Assuming the existence of a simplicial map λ from T into $\diamondsuit^{n-1}$ that is antipodal on the boundary implies a continuous map $||\lambda||$ from B^n into S^{n-1} that is antipodal on the boundary, contradicting (BU2b).

To prove the converse implication, which is our primary interest, assume $f : B^n \to S^{n-1}$ is a continuous map that is antipodal on the boundary. We construct a triangulation T of B^n and a map λ contradicting the theorem above. Here, T can be chosen as any triangulation of B^n that is antipodal on the boundary and has simplex diameter at most δ, where δ is specified as $\varepsilon := \frac{1}{\sqrt{n}}$.

A continuous function on a compact set is uniformly continuous, ensuring the existence of $\delta > 0$ such that if the distance between two points $x, x' \in B^n$ does not exceed δ, then $\|f(x) - f(x')\|_\infty < 2\varepsilon$. This δ bounds the diameter of the simplices in T.

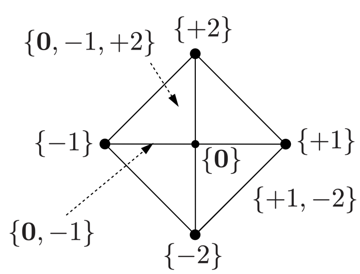
We can define $\lambda : V(T) \to \pm 1, \pm 2, \pm 3, ..., \pm n$.

$$k(v) = \min\{i : |f(v)_i| \geq \varepsilon\}$$

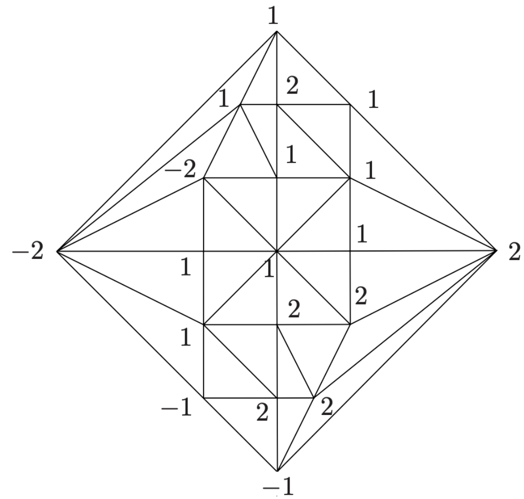
$$k(v) = \begin{cases} +\,k(v) & \text{if } f(v)_{kv} > 0 \\ -\,k(v) & \text{if } f(v)_{kv} < 0 \end{cases}$$

Because f is antipodal on ∂B^n, it follows that $\lambda(-v) = -\lambda(v)$ for every vertex v on the boundary. Therefore, Tucker's lemma is applicable, resulting in the existence of a complementary edge vv'. Let $i = \lambda(v) = -\lambda(v') > 0$. Consequently, $f(v)_i \geq \varepsilon$ and $f(v')_i \leq -\varepsilon$, leading to $\|f(v) - f(v')\|_\infty \geq 2\varepsilon$, which contradicts our earlier assumptions.

Proof. We begin by setting specific conditions for the triangulation T. Initially, we substitute the Euclidean ball B^n with the crosspolytope $\hat{B}^n$, defined as the unit ball under the ℓ_1-norm. Let $\diamondplus^n$ denote the standard triangulation of $\hat{B}^n$ induced by the coordinate hyperplanes. Explicitly, each simplex $\sigma \in \diamondplus^n$ either resides within $\Diamond^{n-1}$ (these are the simplices on the boundary), or consists of a union $\tau \cup \{0\}$ where $\tau \in \Diamond^{n-1}$; in essence, it forms a cone with base σ and apex at the origin. The illustration below depicts $\diamondplus^2$, highlighting certain simplices by their sets of vertices.



The second requirement specifies that the sign of each coordinate remains consistent across the interior of every simplex $\sigma \in T$. We refer to such a triangulation T as a specialized triangulation of $\hat{B}^n$. For $n = 2$, a specialized triangulation T with a labeling λ as described in Tucker's lemma is illustrated below:



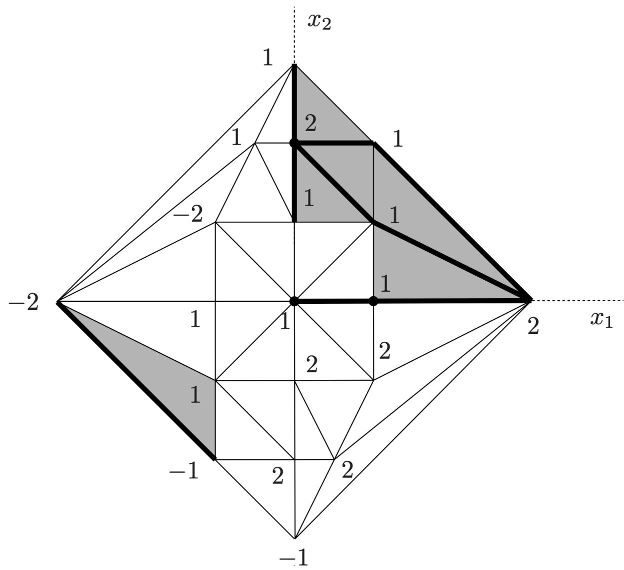
Therefore, we assume that T is a specialized triangulation of $\hat{B}^n$ and $\lambda : V(T) \to \{\pm 1, \pm 2, \ldots, \pm n\}$ is a labeling that is antipodal on the boundary. The proof employs a parity argument, albeit not a straightforward one; it necessitates consideration of simplices of all possible dimensions. We identify a subset of simplices within T, termed "happy" simplices, where λ exhibits a specific behavior. These simplices are organized into a graph, and we demonstrate a contradiction by showing that this graph contains exactly one vertex with an odd degree.

For a simplex $\sigma \in T$, define $\lambda(\sigma) := \{\lambda(v) : v \text{ is a vertex of } \sigma\}$. Additionally, we define another set $S(\sigma)$ of labels (independent of λ values on σ): choose a point x in the relative interior of σ, then set

$$S(\sigma) := \{+i : x_i > 0,\ i = 1, 2, \ldots, n\} \cup \{-i : x_i < 0,\ i = 1, 2, \ldots, n\}.$$

Since T is a specialized triangulation, all choices of x yield the same $S(\sigma)$. Geometrically, $S(\sigma)$ represents the vertex set of the simplex in $\diamondsuit^{n-1}$ to which σ maps under central projection from 0 (where the "exceptional" simplices $\emptyset$ and $\{0\}$ map to $\emptyset$).

A simplex $\sigma \in T$ is termed "happy" if $S(\sigma) \subseteq \lambda(\sigma)$. In other words, $S(\sigma)$ serves as the set of "desired labels" for σ, and σ qualifies as happy if all these labels are indeed present on its vertices. The following illustration highlights the happy simplices:



First, we explore certain characteristics of the happy simplices. Consider σ as a happy simplex, where $k = |S(\sigma)|$. σ resides in the k-dimensional linear subspace L_σ, which is spanned by the coordinate axes x_i such that $i \in S(\sigma)$ or $-i \in S(\sigma)$. Therefore, dim $\sigma \leq k$. Conversely, dim $\sigma \geq k - 1$, since at least k vertex labels are required for σ to be happy. We classify σ as tight if dim $\sigma = k - 1$, meaning all vertex labels are essential for its happiness. Otherwise, if dim $\sigma = k$, σ is termed loose. In the case of a loose happy simplex σ, either some vertex label repeats or an extra label not in $S(\sigma)$ exists.

A boundary happy simplex must be tight, while a non-boundary happy simplex can be either tight or loose. The simplex $\{0\}$ is consistently happy and always considered loose.

We define an undirected graph G where vertices represent all happy simplices. Two vertices σ and $\tau \in T$ are connected by an edge if:

Conditions

(1) σ and τ are antipodal boundary simplices ($\sigma = -\tau \subset \partial \hat{B}^n$); or
(2) σ is a facet of τ (i.e., a dim $\tau - 1$-dimensional face) with $\lambda(\sigma) = S(\tau)$; this implies that the labels on σ alone suffice to make τ happy.

The simplex $\{0\}$ has a degree of 1 in G, connected precisely to the edge of the triangulation made happy by the label $\lambda(0)$. We establish that if there is no complementary edge, any other vertex σ in the graph G has a degree of 2. Since a finite graph cannot have only one vertex with an odd degree, this conclusion verifies Tucker's lemma.

We distinguish between several scenarios:

(1) If σ is a tight happy simplex: - Any neighboring simplex τ of σ is either $-\sigma$ or has σ as a facet. There are two subcases:
   (a) If σ lies on the boundary $\partial \hat{B}^n$, then $-\sigma$ is one of its neighbors. Any other neighbor τ must have σ as a facet to satisfy its labels. Hence, τ lies in the k-dimensional coordinate subspace L_σ, where $k = \dim \sigma + 1$. The intersection $L_\sigma \cap \hat{B}^n$ forms a k-dimensional crosspolytope, and the simplices of T contained in L_σ triangulate this space. If σ is a boundary $(k-1)$-dimensional simplex in a triangulation of $\hat{B}^k$, it serves as a facet of exactly one k-simplex.
   (b) If σ does not lie on the boundary, similar reasoning shows that σ is a facet of exactly two simplices made happy by its labels, which are its two neighbors.

(2) If σ is a loose happy simplex: The subcases are:
   (a) If $S(\sigma) = \lambda(\sigma)$, implying one label occurs twice on σ, then σ is adjacent to exactly two of its facets (and cannot be a facet of a happy simplex).
   (b) If there is an extra label $i \in \lambda(\sigma) \setminus S(\sigma)$, note that $-i \notin S(\sigma)$, otherwise there would be a complementary edge. One of σ's neighbors is the facet of σ that does not include the vertex with the extra label i. Moreover, σ is a facet of exactly one loose simplex σ' made happy by the labels of σ, specifically σ' with $S(\sigma') = \lambda(\sigma) = S(\sigma) \cup \{i\}$. We encounter σ' by moving from an interior point of σ towards the $x_{|i|}$-axis, positively for $i > 0$ and negatively for $i < 0$.

In each case, we establish that σ has exactly two neighbors, leading to a contradiction.

III Applications of Borsuk-Ulam

III.I Ham Sandwich Theorem

This theorem claims that every sandwich containing ham and cheese can be cut by a singular planar cut, in such a way that both pieces will contain equal amounts of ham, cheese, and bread.

Theorem 6 Given n measurable sets $A_1, A_2, \ldots, A_n$ in $\mathbb{R}^n$, each with finite measure (volume), there exists a hyperplane in $\mathbb{R}^n$ that simultaneously bisects the volume of each of these sets into two equal halves.

In other words, there exists a hyperplane H such that each of the n sets $A_1, A_2, \ldots, A_n$ intersects H in such a way that the volume of $A_i \cap H^+$ (the part of A_i on one side of H) equals the volume of $A_i \cap H^-$ (the part of A_i on the opposite side of H), for $i = 1, 2, \ldots, n$.

Proof. Consider each measurable set $A_i \subset \mathbb{R}^n$ as its characteristic function $\chi_{A_i} : \mathbb{R}^n \to \{0, 1\}$. This function is 1 inside A_i and 0 outside. Now, we define a continuous function $f : S^{n-1} \to \mathbb{R}^n$, where S^{n-1} is the $(n-1)$-dimensional sphere (the boundary of the n-dimensional ball B^n), by:

$$f(x) = \left(\int_{\mathbb{R}^n} \chi_{A_1}(y) x(y) \, dy, \int_{\mathbb{R}^n} \chi_{A_2}(y) x(y) \, dy, \ldots, \int_{\mathbb{R}^n} \chi_{A_n}(y) x(y) \, dy \right)$$

Here, $x \in S^{n-1}$ represents a point on the sphere S^{n-1}, and $x(y)$ denotes the coordinates of x in $\mathbb{R}^n$. The vector $f(x)$ represents the weighted centroid or center of mass of the sets $A_1, A_2, \ldots, A_n$ when each is weighted by the vector x.

According to the Borsuk-Ulam Theorem, there exists a point $x \in S^{n-1}$ such that $f(x) = f(-x)$. This implies that the vector $f(x)$, which is the weighted centroid of the sets A_i, is equal to $f(-x)$, the weighted centroid with $-x$.

The hyperplane H corresponding to x (and thus $-x$) is the hyperplane perpendicular to x (and $-x$) passing through the origin. By the definition of f, this hyperplane H bisects each set A_i into two parts of equal measure:

(1) $A_i \cap H^+$ (the part of A_i on one side of H) has the same measure as
(2) $A_i \cap H^-$ (the part of A_i on the opposite side of H).

This balance holds because $f(x) = f(-x)$ ensures that the weighted centroids on each side of H are equal.

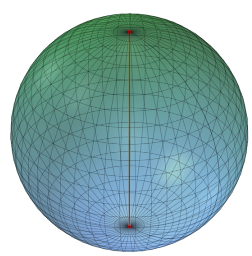
Therefore, the existence of such a hyperplane H that simultaneously bisects each set A_i into two equal parts follows directly from the Borsuk-Ulam Theorem. The theorem guarantees the existence of antipodal points x and $-x$ on S^{n-1} such that $f(x) = f(-x)$, which in turn defines the hyperplane H through their perpendicular bisector.

III.II Necklace Problem

In this problem, two thieves have stolen a valuable necklace with various types of jewels, each type having an even number. Their goal is to divide each type of jewel evenly between them. The challenge is that they must achieve this by partitioning the necklace into contiguous segments and distributing these segments between themselves.

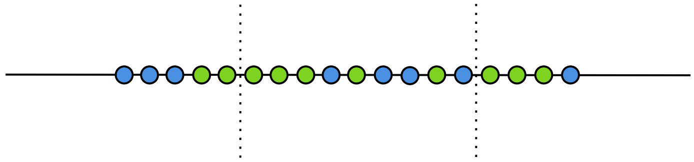
First, let's take a a sphere. A sphere is points in 3D space, represented with three coordinates, so just all possible triplets of numbers. The simplest sphere to describe with coordinates is the standard unit sphere centered at the origin, the set of all points a distance 1 from the origin, meaning all triplets of numbers so that the sum of their squares is 1. So the geometric idea of a sphere is related to the algebraic idea of a set of positive numbers that add up to 1.

If you have one of these triplets, the point on the opposite side of the sphere, the corresponding antipodal point, is whatever you get by flipping the sign of each coordinate.



So, for any function that takes in points on the sphere, triplets of numbers who square sum to 1, and projects some point in 2D space, some pair of coordinates like temperature and pressure, as long as the function is continuous, there will be some input so that flipping all of its signs doesn't change the output.

Now, we can look at the necklace problem. The main problem is that the necklace problem is discrete. so our first step is to translate the stolen necklace problem into a continuous version. We can take this necklace here, with 8 sapphires, and 10 emeralds, and attempt to make a fair division using two cuts.



By making these cuts, we can give one thief the first and last segment, and one thief the middle segment. To make this problem continuous, think of the necklace as a line with length 1, with the jewels sitting evenly spaced on it, and divide up that line into 18 evenly sized segments, one for each jewel. So 8/18ths is sapphire, and 10/18ths is emerald.

Each thief needs to receive 4/18ths sapphire, and 5/18ths emerald. What is important is that if solving the continuous version is possible, so is solving the discrete version. To see this, we imagine doing a fair division where the cuts did not fall evenly between the jewels. Since it's a fair division, even if it cuts unevenly on one side, it cuts unevenly on the other, and in such a way that the total length adds up to a whole number. To make these cuts, you have to choose two places.

Alternatively, we can imagine this as choosing three positive numbers which add up to one. Any time you find three positive numbers that add up to one, it gives you a way to cut the necklace, and vice versa. Following this, we choose which pieces go to which thief.

For any point x, y, z on the sphere, because $x^2 + y^2 + z^2 = 1$, we can cut the necklace so that the first piece has a length x^2, the second has a length y^2, and the third has a length z^2. For that first piece, if x is positive, give it to thief 1, otherwise give it to thief 2. For the second piece, if y is positive, give it to thief 1, otherwise give it to thief 2, and likewise give the third piece to thief 1 if z is positive, and to thief 2 if z is negative. Any way that you divide up the necklace and divvy up the pieces gives us a unique point on the sphere.

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