A Proof and Foundations of the Gauss-Bonnet Theorems in Differential Geometry & Algebraic Topology

Prashanth Prabhala

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1 Abstract

This paper explores the Gauss Bonnet theorem and the linkage between differential geometry and algebraic topology. In this paper, we start off by diving into the mathematical fields of differential and Riemannian geometry, algebraic topology, multivariable calculus, and linear algebra. After covering the required concepts and other additional background definitions, we explore various forms of the Gauss-Bonnet theorem, including the classical version for surfaces without boundary, the extended version for surfaces with smooth and piecewise-smooth boundaries, and advanced generalizations like the Chern-Gauss-Bonnet theorem. I then prove the theorem and examine different versions of the proof for theorem variations. Through rigorous mathematical exposition and illustrative examples, this study elucidates the foundational concepts and also provides a comprehensive understanding of the theorem's applications in mathematics and physics, ranging from the topology of Riemann surfaces to the intricate geometries in general relativity. This paper offers a thorough examination of the Gauss-Bonnet theorem, its' background, proof, and implications, underscoring its importance in mathematics and research.

2 Introduction

Named after the German mathematician Carl Friedrich Gauss (1777–1855) and the French mathematician Pierre Ossian Bonnet (1819–1892), the Gauss-Bonnet theorem honors both of their contributions. Gauss was the first to formulate the theorem in his work *"Disquisitiones Generales Circa Superficies Curvas"* (1827). In the mid-19th century, it was Bonnet who first published the generalized version of the result.

3 Differential Geometry & Topology

The definitions of this chapter are based on Do Carmo [14]. Some concepts are from [24] and [25].

3.1 Regular Surfaces in $\mathbb{R}^3$

A **regular surface** in $\mathbb{R}^3$ is a set $S \subset \mathbb{R}^3$ such that for every point $p \in S$, there exists a neighborhood V of p within S and a smooth map $X : U \subset \mathbb{R}^2 \to V$, where U is an open subset of $\mathbb{R}^2$, satisfying the following conditions:

1. The map $X : U \to V$ is a homeomorphism.
2. For every point $q = (u, v) \in U$, the partial derivatives $\frac{\partial X}{\partial u}$ and $\frac{\partial X}{\partial v}$ are linearly independent.

This means that S can locally be described by smooth parametric equations in two variables, and the parameterization map X preserves the structure and topology of the neighborhood V. The condition on the partial derivatives ensures that X maps the open set U in a way that retains local flatness and differentiability, indicating that S is a smooth surface without self-intersections or singularities in the neighborhood of any point p.

Example 1

Any graphical surface, A, is a regular surface. We consider the map $X : U \to A$ such that $X(u, v) = (u, v, f(u, v))$ for some smooth function f. X is a homeomorphism. Taking the partial derivatives gives:

$$\frac{\partial X}{\partial u} = \left(1, 0, \frac{\partial f(u,v)}{\partial u}\right)$$
$$\frac{\partial X}{\partial v} = \left(0, 1, \frac{\partial f(u,v)}{\partial v}\right)$$

These are linearly independent, so by the definition, any graphical surface A is also a regular surface.

Example 2

The unit sphere, $\mathbb{S}^2$, is a regular surface. A popular parameterization of the unit sphere is given by $X(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$ for $u \in [0, \pi), v \in [0, 2\pi)$. Taking the partial derivatives we get:

$$X_u = (\cos u \cos v, \cos u \sin v, -\sin u)$$

$$X_v = (-\sin u \sin v, \sin u \cos v, 0)$$

Again, these are linearly independent, since $-\sin u < 0$, so by the definition, the unit sphere is a regular surface.

Note: The notation X_u will be used to signify the partial derivative $\frac{\partial X}{\partial u}$.

3.2 Differentiable on a Regular Surface

A real-valued function $f : V \subset S \to \mathbb{R}$ is said to be differentiable at a point $p \in V$ if there exists a parameterization $X : U \subset \mathbb{R}^2 \to S$ such that $p \in X(U) \subset V$ and the composed function $f \circ X : U \to \mathbb{R}$ is differentiable at $X^{-1}(p)$. If this condition holds for every point $p \in V$, then the function f is considered differentiable on V.

The tangent plane $T_p(S)$ at a point p on the surface S is the subspace $dX_p(\mathbb{R}^2)$ consisting of tangent vectors to S at $X(p)$.

3.3 Differential of a Map Between Regular Surfaces

Let S_1 and S_2 be regular surfaces and $f : S_1 \to S_2$ a smooth map. The differential of f at a point $p \in S_1$ is the map

$$df_p : T_p(S1) \to T_{f(p)}(S2)$$

such that for any $v \in T_p(S1)$, and any curve $\alpha : (-\epsilon, \epsilon) \to S1$ with $\alpha(0) = p$ and $\alpha'(0) = v$, if $\beta = f \circ \alpha$, then

$$df_p(v) = \beta'(0) \in T_{f(p)}(S2).$$

3.4 The First Fundamental Form

The first fundamental form of a regular surface S at a point p is the inner product $I_p : T_pS \times T_pS \to \mathbb{R}$ defined by

$$I_p(u, v) = \langle u, v \rangle_{\mathbb{R}^3} = \sum_{i=1}^{3} u_i v_i.$$

3.5 Unit Normal

Given a parameterization $X : U \subset \mathbb{R}^2 \to S$ of a regular surface, the unit normal at a point $p \in U$ is defined as:

$$N(p) = \frac{X_u \times X_v}{|X_u \times X_v|}$$

If $U \subset S$ is open and there exists a unit normal $N : U \to \mathbb{R}^3$ for each $p \in U$, we call N a differentiable field of unit normals on U.

Not all regular surfaces have a differentiable field of unit normals. The concept of orientation helps distinguish surfaces that possess such a field from those that do not. A regular surface, S, is said to be orientable if it has an associated differentiable field of unit normals, N. Let's expand a bit more on orientable manifolds for better understanding.

3.6 Orientability

A manifold M is said to be orientable if it is possible to consistently define a "direction" or "orientation" across the entire manifold. More formally:

Two-Dimensional Surface: A two-dimensional surface S embedded in $\mathbb{R}^3$ is orientable if there exists a continuous choice of unit normal vector at every point on the surface. This means you can smoothly assign a normal vector pointing in a specific direction (e.g., "outwards" or "inwards") at every point without ambiguity.

Higher-Dimensional Manifold: For an n-dimensional manifold M, orientability can be defined in terms of coordinate charts and transition maps: An n-dimensional manifold M is orientable if there exists an atlas of coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ such that all transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ have a positive determinant of the Jacobian matrix. In other words, the transition maps preserve the orientation of the manifold.

The **Jacobian matrix** is a matrix of all first-order partial derivatives of a vector-valued function. Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, which maps an n-dimensional input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to an m-dimensional output vector $\mathbf{y} = (y_1, y_2, \ldots, y_m)$, the Jacobian matrix J of f is an $m \times n$ matrix where each element J_{ij} is given by the partial derivative of the i-th component of f with respect to the j-th component of $\mathbf{x}$:

$$J_{ij} = \frac{\partial y_i}{\partial x_j}$$

In matrix form, the Jacobian matrix J is:

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

The determinant of the Jacobian matrix, if it is a square matrix (i.e., $m = n$), indicates the local scaling factor of the transformation f. A positive determinant means the orientation is preserved under the transformation.

Example 1

The Möbius strip is a classic example of a non-orientable surface. It can be constructed by taking a rectangular strip of paper, twisting one end by 180 degrees, and then joining the ends together to form a loop.

3.7 Gauss Map

Let S be a regular surface in $\mathbb{R}^3$. The Gauss map, N, is a function that maps each point on the surface S to a corresponding point on the unit sphere S^2 in $\mathbb{R}^3$. Mathematically, this is expressed as:

$$N : S \to \mathbb{S}^2$$

where $\mathbb{S}^2$ represents the unit sphere.

The Gauss map assigns to each point on the surface S the unit normal vector at that point. This normal vector is a point on the unit sphere $\mathbb{S}^2$. In essence, the Gauss map translates the geometric properties of the surface into a mapping on the sphere, providing a way to study the curvature of the surface. The Gauss map is crucial for understanding the Gaussian curvature, which is central to our study of the Gauss-Bonnet theorem.

3.8 Weingarten Map or Shape Operator

The Weingarten map, or shape operator, is a linear map $S_p : T_pS \to T_pS$ defined on the tangent plane T_pS at a point p on a regular surface S. For a given tangent vector $v \in T_pS$, the shape operator is defined by:

$$S_p(v) = -dN_p(v),$$

where N is the Gauss map and dN_p is the differential of the Gauss map at p.

The shape operator S_p encodes the way the surface bends by measuring the rate of change of the unit normal vector as one moves along the surface. It is symmetric and its eigenvalues are the principal curvatures of the surface at p.

3.9 The Second Fundamental Form

The second fundamental form of a regular surface S at a point p is a quadratic form $II_p : T_pS \times T_pS \to \mathbb{R}$ that measures how the surface bends at p. It is defined as:

$$II_p(u, v) = \langle dN_p(u), v\rangle,$$

where dN_p is the differential of the Gauss map at p, $u, v \in T_pS$ are tangent vectors, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$.

In terms of the coefficients of the first and second fundamental forms, if E, F, G are the coefficients of the first fundamental form and e, f, g are the coefficients of the second fundamental form, then the second fundamental form can be written as:

$$II = e\, du^2 + 2f\, dudv + g\, dv^2.$$

3.10 Gaussian Curvature

The Gaussian curvature of a regular surface in $\mathbb{R}^3$ at a point p is formally defined as

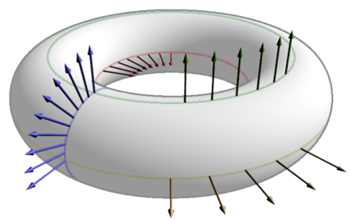
$$K(p) = \det(S(p)), \tag{15}$$

where S is the shape operator and det denotes the determinant.

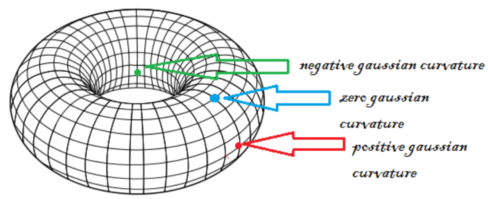
If $x : U \to \mathbb{R}^3$ is a regular patch, then the Gaussian curvature is given by

$$K = \frac{eg - f^2}{EG - F^2}, \tag{16}$$

where E, F, and G are coefficients of the first fundamental form and e, f, and g are coefficients of the second fundamental form [16].



Torus Shape Operator [27]



+, -, & 0 Gaussian Curvature

3.11 Geodesic Curvature

Let $\alpha : [a, b] \to S$ be a curve on the surface S. The geodesic curvature, denoted k_g, is defined as follows:

$$k_g = \frac{\langle \alpha''(t), (N \times \alpha'(t)) \rangle}{|\alpha'(t)|^3}$$

Here, N is the unit normal of the surface.

3.12 Geodesics

Let $\alpha : [a, b] \to S$ be a curve on the surface S. We say that α is geodesic on S if its geodesic curvature $k_g = 0$. Geodesic curvature intuitively measures the

deviation of a curve from being a geodesic. Geodesics are the curves that locally minimize the distance between two points on any given surface or mathematically defined space. Here are a few examples to illustrate geodesics on different surfaces:

Example 1

A simple example is in a standard Euclidean space, the geodesics will be straight lines.

Example 2

The geodesics for a sphere will be its great circles, i.e., the circles obtained from intersecting the sphere with a plane that goes through the centre of the sphere, like the equator or the lines of longitude on planet Earth. These curves represent the shortest path between two points on the spherical surface.

3.13 Regular Regions

3.13.1 Regular Region of a Surface

Let S be a regular surface. We say that a region $R \subset S$ is regular if and only if R is compact and the boundary ∂R is a finite union of piecewise regular curves that do not intersect.

3.13.2 Triangulation of a Regular Region

A triangulation of a regular region R is a finite family T of triangles T_i, $i = 1, 2, \ldots, n$, where by triangle we simply mean a region with three vertices and non-zero external angles, such that

1. $\bigcup_{i=1}^{n} T_i = R$.
2. If $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a common edge of T_i and T_j or a common vertex of T_i and T_j.

3.14 Euler-Poincaré formula

Let T be a triangulation of a regular region $R \subset S$.

The Euler-Poincaré formula is defined as

$$F - E + V = \chi(R)$$

where F is the number of faces of the triangulation, E is the number of sides, and V is the number of vertices. χ is known as the Euler characteristic of the surface S.

The Euler characteristic $\chi(R)$ is a topological invariant that provides a measure of the surface's shape or structure in a way that is independent of the exact geometric form. V, E, and F are the numbers of vertices, edges, and faces, respectively, in the polygonal decomposition of the surface.

3.15 The Gram-Schmidt Process

3.15.1 Vectors

In Euclidean space, a vector is represented as an arrow with a specific magnitude and direction. The arrow starts from one point (called the initial point) and ends at another point (called the terminal point).

Example 1 In two-dimensional space ($\mathbb{R}^2$), a vector can be represented as $\mathbf{v} = (v_1, v_2)$, where v_1 and v_2 are the components of the vector along the x and y axes, respectively.

The Gram-Schmidt process (or procedure) is a sequence of operations that transforms a set of linearly independent vectors into a set of orthonormal vectors that span the same space as the original set.

Let's begin by going over some essential concepts.

- Two vectors r and s are orthogonal if their inner product is zero, i.e., $\langle r, s\rangle = 0$.

- The norm (length) of a vector s given an inner product is defined as $\|s\| = \sqrt{\langle s, s\rangle}$.

- A set of vectors is orthonormal if each vector has unit norm and is orthogonal to each other.

When a basis for a vector space is also an orthonormal set, it is called an orthonormal basis [24].

3.15.2 Steps of the Gram-Schmidt Process [24]

1. Start with a set of linearly independent vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$.

2. Initialize the first orthogonal vector u_1 as the first vector v_1:

$$u_1 = v_1$$

3. For each subsequent vector v_k $(k = 2, 3, \ldots, n)$, orthogonalize it against all previously computed orthogonal vectors:

$$u_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, u_j\rangle}{\langle u_j, u_j\rangle} u_j$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

4. Normalize each orthogonal vector u_k to get an orthonormal set:

$$e_k = \frac{u_k}{\|u_k\|}$$

We have now discussed enough background concepts to proceed to the Gauss-Bonnet theorem.

4 The Gauss-Bonnet Theorems

4.1 Classical Theorem - Surfaces Without Boundary

Let S be a compact and orientable surface in $\mathbb{R}^3$ which lacks a boundary. We can mathematically express the Gauss-Bonnet theorem as:

$$\int_S K\,dA = 2\pi\chi(S)$$

where

- K is the Gaussian curvature.
- dA denotes the area element of the surface.
- $\chi(S)$ is the Euler characteristic of the surface.

Consider a torus or a doughnut-shaped surface (which has no boundary). The Euler characteristic of a torus is 0. The Gauss–Bonnet theorem tells us that the integral of the Gaussian curvature over the entire surface of the torus will also be zero. Now, imagine if you were to stretch and deform the torus in various ways, such as by squeezing it or elongating it. Despite these deformations, the Euler characteristic of the torus would remain 0. Consequently, the total Gaussian curvature of the torus, given by $\int_S K\,dA$, would also remain 0 under such deformations. Now let us generalize this theorem to surfaces with smooth boundary.

4.2 Surfaces With Smooth Boundary

Let M be a compact and orientable surface in $\mathbb{R}^3$ with a smooth boundary ∂M. An example of a surface with smooth boundary is a cylinder. We can mathematically express the Gauss-Bonnet theorem as:

$$\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M)$$

where

- K is the Gaussian curvature.
- k_g is the geodesic curvature of the boundary ∂M.
- dA denotes the area element of the surface.
- ds denotes the line element along the boundary.

- $\chi(M)$ is the Euler characteristic of the surface.

Now, let's further generalize the Gauss–Bonnet theorem to also include surfaces where we only require a piecewise smooth boundary.

4.3 Surfaces With Piecewise-Smooth Boundary

Consider a surface P with only a piecewise smooth boundary and go along its boundary. The Gauss–Bonnet theorem for a piecewise smooth surface is

$$\sum_{\text{vertices}} \omega_i + \int_S K\,dA + \int_{\partial S} k_g\,ds = 2\pi\chi(P)$$

where

- K is the Gaussian curvature.
- k_g is the geodesic curvature of each side ∂P.
- dA denotes the area element of the surface.
- ds denotes the line element along the boundary.
- $\chi(M)$ is the Euler characteristic of the surface.
- ω is the exterior angles at the boundary of the surface.

This theorem is advantageous because it only requires the boundary to be piecewise smooth. Consequently, we can apply the Gauss–Bonnet theorem to structures such as polygons on a plane or polyhedral surfaces.

Other versions of this theorem include the Chern-Gauss-Bonnet theorem and other adapted forms which are discussed in the applications portion of this paper.

5 Proof of the Classical Gauss-Bonnet Theorem

One should note that there are many different versions of the proof for this theorem. In this study, we will prove the classic theorem, the Gauss-Bonnet theorem for surfaces with no boundary. The proof for the other theorem variations can be derived from this. This proof follows the proof by Sigmundur Gudmundsson in [23].

5.1 The Theorem Statement:

Let M be a compact, orientable, and regular C^3-surface in $\mathbb{R}^3$. If K is the Gaussian curvature of M, then

$$\int_M K\,dA = 2\pi \cdot \chi(M)$$

where $\chi(M)$ is the Euler characteristic of the surface.

5.2 Regular C^3-surfaces

A regular C^3-surface refers to a surface that is smooth and has continuous derivatives up to the third order. Here, M is smooth up to the third derivative, which is a common requirement in differential geometry for the analysis of curvature and other geometric properties.

5.3 Green's Theorem

The Green's theorem is a fundamental result in vector calculus that relates a double integral over a two-dimensional region to a line integral around the boundary of that region.

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then Green's theorem states:

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where:

- $\oint_C$ denotes the line integral around the boundary C.
- $\iint_D$ denotes the double integral over the region D.
- P and Q are functions of x and y with continuous partial derivatives.
- dA is the area element in D.

5.4 Geodesic Curvature Expression

Let M be an oriented regular C^3-surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$.

Consider a local parametrization $X : U \to X(U)$ of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \to X(U)$ parametrize a regular, closed, simple, and positively oriented C^2-curve on $X(U)$ by arclength. Let

$\kappa_g : \mathbb{R} \to \mathbb{R}$ be its geodesic curvature. Define the orthonormal basis $\{Z, W\}$ obtained by applying the Gram-Schmidt process (refer to 3.15) on the basis $\{X_u, X_v\}$ from the local parametrization $X : U \to X(U)$.

Along the curve $\gamma : \mathbb{R} \to X(U)$, we define an angle $\theta : \mathbb{R} \to \mathbb{R}$ such that the unit tangent vector $\dot{\gamma}$ satisfies:

$$\dot{\gamma}(s) = \cos\theta(s) \cdot Z(s) + \sin\theta(s) \cdot W(s).$$

Then, for the second derivative $\ddot{\gamma}$, we have:

$$\ddot{\gamma}(s) = \dot{\theta}(s) \cdot (-\sin\theta(s) \cdot Z(s) + \cos\theta(s) \cdot W(s)) + \cos\theta(s) \cdot \dot{Z}(s) + \sin\theta(s) \cdot \dot{W}(s).$$

This implies that the geodesic curvature κ_g satisfies:

$$\kappa_g = \langle N \times \dot{\gamma}, \ddot{\gamma} \rangle = \dot{\theta} - \langle Z, \dot{W} \rangle.$$

where:

- N is the Gauss map.
- Z and W are orthonormal basis vectors.
- θ is the angle between the unit tangent vector $\dot{\gamma}$ and the orthonormal basis $\{Z, W\}$.
- $\langle \cdot, \cdot \rangle$ denotes the inner product.

5.5 Theorem 5.5

Let M be an oriented regular C^3-surface in $\mathbb{R}^3$ with a Gauss map $N : M \to S^2$.

Consider a local parametrization $X : U \to X(U)$ of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \to X(U)$ parameterize a positively oriented, simple, piecewise regular C^2-polygon on M by arclength. Let $\text{Int}(\gamma)$ denote the interior of γ and let $\kappa_g : \mathbb{R} \to \mathbb{R}$ be its geodesic curvature on each regular piece. If $L \in \mathbb{R}^+$ is the period of γ, then:

$$\int_0^L \kappa_g(s)\, ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \int_{\text{Int}(\gamma)} K\, dA.$$

Here, K is the Gaussian curvature of M and $\alpha_1, \ldots, \alpha_n$ are the interior angles at the n corner points [23].

5.6 Proof of Theorem 5.5

Let $\{Z, W\}$ be the orthonormal basis obtained by applying the Gram-Schmidt process on the basis $\{X_u, X_v\}$, derived from the local parametrization $X : U \to X(U)$ of M. Let D be the discrete subset of $\mathbb{R}$ corresponding to the corner points of $\gamma(\mathbb{R})$. Along the regular arcs of $\gamma : \mathbb{R} \to X(U)$, we define an angle $\theta : \mathbb{R} \setminus D \to \mathbb{R}$ such that the unit tangent vector $\dot{\gamma}$ satisfies:

$$\dot{\gamma}(s) = \cos\theta(s) \cdot Z(s) + \sin\theta(s) \cdot W(s).$$

We have earlier seen that in this case the geodesic curvature is given by $\kappa_g = \dot{\theta} - \langle Z, \dot{W} \rangle$ and integration over one period gives:

$$\int_0^L \kappa_g(s)\, ds = \int_0^L \dot{\theta}(s)\, ds - \int_0^L \langle Z(s), \dot{W}(s) \rangle\, ds.$$

Using Green's theorem (refer to 5.3), we have:

$$\int_0^L \langle Z(s), \dot{W}(s) \rangle\, ds = \int_{\mathrm{Int}(\gamma)} K\, dA.$$

The integral over the derivative $\dot{\theta}$ splits into integrals over each regular arc:

$$\int_0^L \dot{\theta}(s)\, ds = \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \dot{\theta}(s)\, ds.$$

This measures the change of angle with respect to the orthonormal basis $\{Z, W\}$ along each arc. At each corner point, the tangent jumps by the angle $(\pi - \alpha_i)$ where α_i is the corresponding inner angle. When moving around the curve once, the changes along the arcs and the jumps at the corner points add up to 2π.

Hence:

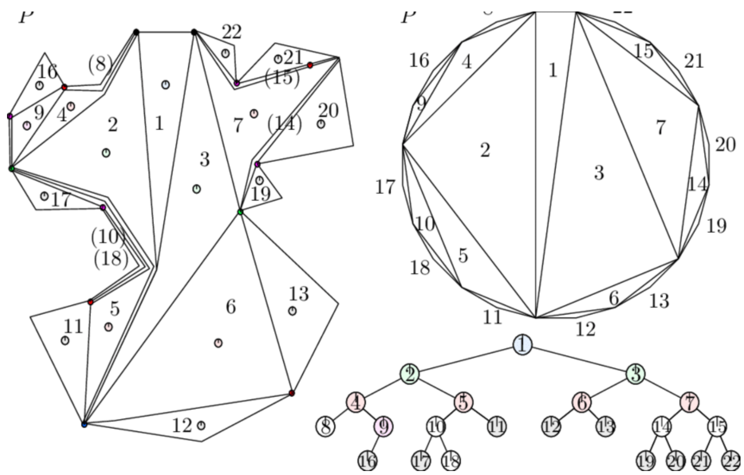
$$2\pi = \int_0^L \dot{\theta}(s)\, ds + \sum_{i=1}^{n} (\pi - \alpha_i).$$

This proves the statement made in theorem 5.4.

5.7 Proof of the Gauss-Bonnet theorem

A geodesic polygon is a polygon on a curved surface where each side is a geodesic segment. Recall our definition of geodesics in section 3.12.

Let $T = \{T_1, \ldots, T_F\}$ be a division of the surface M such that each T_k is a geodesic polygon contained in the image $X_k(U_k)$ of a local parametrisation $X_k : U_k \to X_k(U_k)$ of M. Then the integral of the Gaussian curvature K over M splits as follows:



Geodesic triangulations [26]

$$\int_M K\,dA = \sum_{k=1}^{F} \int_{T_k} K\,dA$$

into the finite sum of integrals over each polygon $T_k \in T$. According to the theorem in Section 5.5, we now have:

$$\int_{T_k} K\,dA = \sum_{i=1}^{n_k} \alpha_{ki} + (2 - n_k)\pi$$

for each of the geodesic polygons T_k. By adding these relations we then obtain:

$$\begin{aligned}\int_M K\,dA &= \sum_{k=1}^{F} \left((2 - n_k)\pi + \sum_{i=1}^{n_k} \alpha_{ki} \right) \\ &= 2\pi F - 2\pi E + \sum_{k=1}^{F} \sum_{i=1}^{n_k} \alpha_{ki} \\ &= 2\pi(F - E + V)\end{aligned}$$

This proves the Gauss-Bonnet theorem for surfaces with no boundary.

6 Applications

6.1 Mathematical Applications:

This fundamental result in differential geometry, has profound applications in pure mathematics. In this study, we discuss 4 specific mathematical applications with the theorem and expanded versions of it.

6.1.1 The Chern-Gauss-Bonnet theorem

The Chern-Gauss-Bonnet theorem extends the Gauss-Bonnet theorem to higher-dimensional manifolds, relating the topology of a manifold to its curvature through characteristic classes, which are used to study vector bundles and their properties. The theorem states:

$$\int_M \mathrm{Pf}(R) = (2\pi)^{n/2}\chi(M),$$

where

- $\mathrm{Pf}(R)$ is the Pfaffian of the curvature form R.
- M is an even-dimensional compact orientable manifold.

This theorem is crucial in characteristic classes and index theory, forming the basis for results like the Atiyah-Singer Index Theorem.

6.1.2 Spectral Geometry & The Hodge Theorem

Moreover, the Gauss-Bonnet theorem finds applications in the study of spectral geometry, where it extends to a broader class of spectral triples, demonstrating its versatility in diverse mathematical contexts. This extension highlights the theorem's capacity to provide insights into the geometric and spectral properties of various mathematical structures. The theorem serves as a foundational tool for proving essential results, such as the Hodge Theorem. The Hodge Theorem states:

$$\Delta\omega = 0 \quad \Leftrightarrow \quad \omega = d\alpha + \delta\beta + \gamma$$

where

- Δ is the Laplace operator.
- ω is a differential form.
- d is the exterior derivative.
- δ is the codifferential.
- γ is the harmonic component of ω.

This theorem is crucial for decomposing differential forms and connecting the topology of a manifold with its geometry.

6.1.3 Moduli Spaces of Riemann Surfaces

In algebraic geometry, moduli spaces are geometric spaces that parameterize a class of objects, such as algebraic curves or Riemann surfaces, up to an equivalence relation. The Gauss-Bonnet theorem has been instrumental in deriving formulas for these moduli spaces, offering insights into their geometric structure. For example, the theorem can be used to compute the Euler characteristic of moduli spaces of Riemann surfaces. This application is crucial because it connects the geometric properties of the surfaces (through curvature) to topological invariants (like the Euler characteristic), facilitating the study of complex geometric objects.

The theorem's relevance in this context lies in its ability to relate local geometric properties (curvature) to global topological features, which is essential for understanding the overall shape and structure of the moduli spaces.

6.1.4 Finsler Geometry

In the study of Finsler geometry, the theorem forms the basis for establishing Gauss-Bonnet-Chern theorems for complex Finsler manifolds. This application highlights the theorem's significance in non-Riemannian geometries and its ability to generalize to complex geometric settings. The Gauss-Bonnet-Chern theorem for Finsler manifolds can be stated as:

$$\int_M K_F \, dV_F = 2\pi\chi(M),$$

where

- K_F is the flag curvature in Finsler geometry, and
- dV_F is the volume form associated with the Finsler structure.

By exploring the implications of the Gauss-Bonnet theorem in Finsler geometry, mathematicians have been able to derive essential results that deepen our understanding of geometric structures beyond traditional Riemannian settings.

6.2 Physics Applications:

The Gauss-Bonnet theorem also finds extensive applications in various branches of physics. In this study, we talk about 2 specific applications in general relativity and black hole thermodynamics.

6.2.1 General Relativity

In general relativity, the fabric of spacetime is described by a four-dimensional Lorentzian manifold, whose curvature is determined by the Einstein field equations. The Gauss-Bonnet theorem, originally formulated for two-dimensional surfaces, has been extended to higher dimensions and plays a crucial role in understanding the interplay between curvature and topology. The Gauss-Bonnet theorem in its higher-dimensional form is part of the Gauss-Bonnet-Chern theorem, which relates the integral of the curvature of a manifold to its Euler characteristic, a topological invariant. For a four-dimensional Riemannian manifold, the theorem is expressed as:

$$\int_{\mathcal{M}} \left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \right) \, dV = 32\pi^2 \chi(\mathcal{M})$$

- R_{abcd} is the Riemann curvature tensor.
- R_{ab} is the Ricci curvature tensor.
- R is the scalar curvature,
- $\chi(\mathcal{M})$ is the Euler characteristic of the manifold $\mathcal{M}$.
- dV is the volume element of the manifold.

This relationship shows how the integral of certain curvature invariants over the entire manifold is related to a purely topological quantity, the Euler characteristic. In general relativity, this helps in understanding how the curvature induced by the presence of mass and energy (as described by the Einstein field equations) affects the global properties of spacetime.

The implications of the Gauss-Bonnet theorem for spacetime solutions are profound. For instance, it provides insight into the global topology of the Schwarzschild and Kerr solutions, which describe non-rotating and rotating black holes, respectively. It is essential for studying the spatial behavior of the Gauss-Bonnet curvature invariant of rapidly-rotating Kerr black holes. By analyzing the physical and mathematical properties of the Gauss-Bonnet curvature invariant in the vicinity of spinning black holes, researchers have uncovered nontrivial behaviors that provide insights into the dynamics of black hole spacetimes [3].

6.2.2 Black Hole Thermodynamics

Black hole thermodynamics is a field that explores the analogy between the laws of thermodynamics and the properties of black holes. In the realm of higher-dimensional theories of gravity, the Gauss-Bonnet curvatures are utilized to describe gravity in scenarios beyond the standard four dimensions. The curvatures play a significant role in theories such as Lovelock gravity, Gauss-Bonnet Gravity, and Lanczos gravity, providing a framework to understand gravitational interactions in higher-dimensional spaces [6]. The Gauss-Bonnet term is the second-order term in the Lovelock series and is given by:

$$\mathcal{L}_{GB} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

The action incorporating the Gauss-Bonnet term in D-dimensional spacetime is:

$$S = \int d^D x\sqrt{-g}\,(R + \alpha\mathcal{L}_{GB} + \cdots)$$

where

- α is a coupling constant.

The entropy of a black hole in higher dimensions can be computed using the Wald entropy formula, which incorporates the contributions from higher-order curvature terms. For a black hole solution in a theory with the Gauss-Bonnet term, the entropy S is given by:

$$S = \frac{A}{4G}\left(1 + \frac{2\alpha}{(D-3)(D-4)}R_H\right)$$

where

- A is the area of the event horizon.
- G is the gravitational constant.
- R_H is the Ricci scalar evaluated on the horizon.

The Gauss-Bonnet term thus modifies the usual Bekenstein-Hawking entropy formula by adding a correction term dependent on the curvature of the horizon. This modified entropy expression influences the stability and phase transitions of black holes in higher-dimensional spacetimes, leading to richer thermodynamic behavior compared to four-dimensional black holes.

Furthermore, in the context of black hole physics, the theorem is linked to the optical Berry phase and the emergence of black hole entropy through dimensional continuation [1]. By exploring the inner structure of the Gauss-Bonnet-Chern theorem and its connection to the optical Berry phase, researchers have uncovered deep relationships between geometric properties and physical phenomena associated with black holes.

7 Conclusion

The Gauss-Bonnet theorem stands as an elegant theorem, bridging the powerful fields of algebraic topology and differential geometry.

Through various proofs and interpretations, the Gauss-Bonnet theorem continues to reveal the deep connections between geometry and topology, demonstrating how local geometric properties can dictate global topological outcomes. Its influence spans multiple disciplines, proving its timeless relevance and ongoing utility in advancing mathematical and physical theories.

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