Model Theory

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What is Logic

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The simplest system of logic that we can work with is called propositional logic. Here we work with propositions, sentences which are either true or false, connected by connectives.

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Similarly, the other connectives include $\neg$ (negation), $\vee$ (or), $\implies$ (implication) and $\leftrightarrow$ (bi-implication).

First Order Logic

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Thus, we work with a stronger system of logic known as **first order logic**. Here we have terms (objects we're talking about) and formulas (statements about those objects).

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Syntax is about deriving true statements from true statements using a formal set of rules called deduction rules. If we have the proposition $P = A \vee B$, and the proposition $Q = \neg B$, the deduction rule for $\vee$ allows us to deduce A from P and Q. We denote syntactic equivalence by the symbol $\vdash$. $A \vdash B$ means that there exists a proof of A from B.

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Semantics is about the truth valuations of statements. Semantics is about the **truth valuation** of sentences, and can be visualized through truth tables.

p	q	p$\wedge q$	(p$\wedge q) \to p$	p$\vee q$	p$\to (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	T

Syntax vs. Semantics

We say A and B are semantically equivalent if A and B have the same truth table. Semantic entailment is denoted by the symbol $\vDash$.

A History of Model Theory

Over the past few years, logic has broadly been divided into proof theory and model theory. While proof theory reasons on the side of the syntax, model theory is about the semantic side of logic. It is less about formal derivations and closer in spirit to classical mathematics, inspiring the comment in Van Dalen's popular textbook **'Logic and Structure'**- "If proof theory is about the sacred, then model theory is about the profane."

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The term **Model Theory** goes back to Alfred Tarski, who first used the term "Theory of Models" in a publication in 1954.

Since the 1970s, the subject has been shaped by Saharon Shelah's stability theory. It has also been of increasing interest to mathematicians, and been useful in proving a number of theorems across different areas of mathematics, particularly algebraic and Diophantine geometry.

The Language of First-Order Logic- Logical Symbols

The **logical** symbols consist of the following-

- The equality symbol $=$.
- The connectives- These are $\neg$ (negation), $\wedge$ (and), $\vee$ (or), $\implies$ (implication), $\iff$ (bi-implication), $\forall$, (for all), and $\exists$ (there exists). These function like the words in everyday language they correspond to.
- The variables- v_0, v_1,...,v_n- We can use as many variable as we want in a formula. Sometimes, we just use the variable x, y, z instead of the variable v_i indexed by the natural numbers.
- Punctuation symbols such as the parentheses (and).

The Language of First-Order Logic- Vocabulary

- Constant symbols: Often denotes by the letter c with subscripts.
- Function symbols: Often denoted by the letter F with subscripts. These are m-placed functions for some natural number m, which means that the function takes m arguments. For instance, $+$ is a 2-placed or binary function symbol.
- Relation symbols: A relation symbol, usually denoted by the letter R with subscripts stands for the n-placed relation R for some natural number n, which means that it takes n arguments.

Examples

A language is denoted as $\mathcal{L} = (c_0, c_1,...,F_0, F_1,...R_0, R_1)$, where $c_0, c_1,...$ are the constants, $F_1, F_2,...$ are the functions, and $R_1, R_2,...$ are the relations in that language. Here are some examples-

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- When talking about groups, we can use the language $(e,\ *,\ ^{-1})$, where e is the neutral element of the group, $*$ is the operation defined on the group, and $^{-1}$ is the inverse function (a function taking only 1 argument).

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- Set theory can be encoded by using the language only containing the relation $\in$, which stands for 'belongs to'.

Terms and Formulas

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A sentence is a formula that has no 'free' variables. For example, $(_i)v_i = 3$ is a sentence, while $v_1 = 3$ is not.
The formal definitions of terms and formulas are given recursively.

What's a Model

Statements in a language don't really mean anything unless we have an **interpretation** for them. Let's say we have a language $\mathcal{L} = (c_1, R_1, F_1)$ where R_1 and F_1 are a 2-placed relation and function symbol respectively. Here, we could interpret c_1 as 0, R_1 as the $>$ symbol, and F_1 as $+$, if we're working with an Abelian group. But we could also interpret c_1 as the empty set, R_1 as the $\in$ relation, and F_1 as $\cup$ if we're working with sets.

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A set with an interpretation for the objects of a language is called a **model**. A model $\mathfrak{U}$ in a language consists of a set $\mathbf{A}$ along with an interpretation function I which maps all the constants, relations and functions of L to elements in $\mathbf{A}$, relations on $\mathbf{A}^n$, and functions from $\mathbf{A}^n$ to $\mathbf{A}$ respectively.

What is a Model

The interpretation function maps relations and functions the way we would expect them to. For instance, if we have a language $L = (c_1, c_2, R_1, F_1, F_2)$, where R_1 is a 2-placed relation symbol and F_1 and F_2 are 2-placed function symbol, we could consider the set **R** of the real numbers, interpret c_1 and c_2 as the special constants 0 and 1 respectively, interpret R_1 as the $\geq$ symbol and F_1 and F_2 as the $+$ and $\cdot$ signs.

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The primary idea of model theory is that we can think of mathematical objects as models of a language.

Some Definitions

We say that $\mathfrak{U} \models \psi$ for a model $\mathfrak{U}$ and sentence ψ if ψ is true for the model $\mathfrak{U}$.

If Σ is a set of sentences, A is said to be a model of Σ, written $\mathrm{A} \models \Sigma$, whenever $A \models \sigma$ for each $\sigma \in \Sigma$. Σ is said to be satisfiable iff there is some A such that $A \models \Sigma$.

A **theory** T is a set of sentences. If T is a theory and σ is a sentence, $T \models \sigma$ if for all models $\mathfrak{U}$, if $\mathfrak{U} \models \mathcal{T}$ then $\mathfrak{U} \models \sigma$.

Model Existence Theorem

A set of sentences is said to be **consistent** if we can't deduce a contradiction from them. Note that here we're reasoning on the side of the syntax- we're implying that one can't apply the logical rule of proofs to deduce a contradiction from the set of sentences.

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The **Model Existence Theorem** states that every set of consistent sentences has a model. (If $\Sigma \nvdash \perp$, then there exists a model $\mathfrak{U} \models \Sigma$.)

This is equivalent to Godel's famous completeness theorem for first order logic.

Godel's Completeness Theorem

Consider a set of sentences Σ and a sentence ψ such that $\Sigma \vDash \psi$.

For any model $\mathfrak{U}$, $\mathfrak{U} \vDash \Sigma \implies \mathfrak{U} \vDash \psi$.

Thus, $\Sigma \cup \neg\psi$ has no models.

Then, by the contrapositive of the Model Existence Theorem, $\Sigma \cup (\neg\psi) \vdash \bot$.

By the deduction theorem, $\Sigma \vdash (\neg\psi \implies \bot)$.
From here we get $\Sigma \implies \psi$

Compactness Theorem

Let Σ be any set of formulas. Σ has a model if and if and only if every finite subset of Σ has a formula.

The $\rightarrow$ direction is by trivial. To prove the $\rightarrow$ direction, we assume that Σ has no model and then show that some finite $\Sigma_0 \in \Sigma$ doesn't have a model.

There exist non-standard models of arithmetic

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The language of natural numbers is traditionally (N, $+$, $\cdot$, 0, 1). We add a constant c to the language L to get (N, $+$, $\cdot$, 0, 1) $\cup$ c. Along with the standard axioms of Peano arithmetic, we include the infinite set of axioms c ¿ n for each natural number n. We know that this theory has models by the compactness theorem.

Any finite subset of these axioms is satisfied by a model that is the standard model of arithmetic plus the constant x interpreted as some number larger than any numeral in the finite subset.

Finite and Infinite Models

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Model Theory can also allow us to go from the infinite to the finite.
Ramsey's Theorem: For each $n \in \mathbb{N}$, there is an $r \in \mathbb{N}$ such that if $\mathfrak{G}$ is any graph with r vertices, then either $\mathfrak{G}$ contains a complete subgraph with n vertices or a discrete subgraph with n vertices. Ramsey began by proving an infinite version of the theorem, which can then be adapted for finite graphs using model theory.

Some Other Applications

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- The existence of **non-standard analysis**, based on infinitesimals (numbers that are smaller than any real number) is shown to be possible by model theory. It provides an alternate way for doing calculus in contrast to epsilon-delta definitions.
- By assuming that the theory of RCF (real closed fields, like the reals), we can provide a solution to **Hilbert's 17th problem** from his famous list of 23 problems for the 20th century. The problem asks if given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions.

References

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