MODEL THEORY

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1. Historical Introduction

Model Theory is a branch of logic which is primarily concerned with theories, which are collection of true sentences, and their models, that is structures for which the particular theory holds true.

There's traditionally a contrast between **syntax** and **semantics** in logic. To some extent, model theory reasons about logic on the side of the semantics. This is in contrast to proof theory, the other major area of logic, which is primarily syntactic in nature. Traditionally, model theory has been less concerned with formal rigor and been closer to the classical style of doing mathematics, inspiring the comment in Van Dalen's popular textbook **'Logic and Structure'**- "If proof theory is about the sacred, then model theory is about the profane."

The first instance of model theory can be traced back to Charles Sanders Peirce and Ernst Schroder, when semantics started playing a role in Logic. The term 'model theory' goes back to Alfred Tarski, who first used the term "Theory of Models" in a publication in 1954. Since the 1970s, the subject has been shaped by Saharon Shelah's stability theory. It has also been of increasing interest to mathematicians, and been useful in proving a number of theorems across different areas of mathematics, particularly algebraic and Diophantine geometry.

2. Introduction to the Language of Mathematical Logic

Mathematical logic is the study of developing formal systems to study how we derive true statements from true statements. In this paper we look at first-order model theory which is built upon a system of logic known as first-order logic. First order logic is powerful enough to encode all of mathematics.

Every logic has three essential components- the language, the rules and the axioms. The language is a set of symbols and rules to work with them that allows us to express any statement formally. I start by defining the language of first order logic.

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Definition 2.1 (The Logical Symbols)**.** The logical symbols consist of the following:

- The equality symbol $=$ which signifies two terms being equal.
- The connectives: These are $\neg$ (negation), $\wedge$ (and), $\vee$ (or), $\implies$ (implication), $\iff$ (bi-implication), $\forall$, (for all), and $\exists$ (there exists). These function like the words in everyday language they correspond to.
- The variables: v_0, v_1,...,v_n- We can use as many variable as we want in a formula. Sometimes, we just use the variable x, y, z instead of the variable v_i indexed by the natural numbers.
- Punctuation symbols such as the parentheses '(' and ')'.

These logical symbols are components of **all** languages. However, we use different languages when talking about different areas of math. These different elements that define a language are called the vocabulary of the language. The essential elements of the vocabulary of the language include the constants, relations and functions.

Definition 2.2 (Vocabulary of a Language)**.** The vocabulary of a language includes the following:

- Constant symbols: Often denotes by the letter c with subscripts.
- Function symbols: Often denoted by the letter F with subscripts. These are m-placed functions for some natural number m, which means that the function takes m arguments. For instance, $+$ is a 2-placed or binary function symbol.
- Relation symbols: A relation symbol, usually denoted by the letter R with subscripts stands for the n-placed relation R for some natural number n, which means that it takes n arguments.

We use different languages when dealing with different areas of math, which have different constants, relations and functions. Conventionally, we can define a language as (c_0, c_1,...,F_0, F_1,...R_0, R_1...), where c_0, c_1,... are the constants, F_0, F_1,... are the functions, and R_0, R_1,... are the relations in that language. As all languages have the same logical symbols, a language can be defined unambiguously by stating the elements of it's vocabulary.

A few common examples of languages we might encounter are as follows:

- When trying talking about any ordered ring or field such as the integers, rationals or reals, we might use the language $(0, 1, +, \cdot, \leq)$.

- When talking about groups, we can use the language (e, $*$, $^{-1}$), where e is the neutral element of the group, $*$ is the operation defined on the group, and $^{-1}$ is the inverse function (a function taking only 1 argument).
- Set theory can be encoded by using the language only containing the relation $\in$, which stands for 'belongs to'.

When dealing with languages, we define **terms** and **formulae**. A term essentially refers to a mathematical object in our language that we're talking about. For instance, when talking about natural numbers, any number would be a term. In general, all the constant symbols and variables of a language of a language are terms, and so are any functions that take other terms as arguments. Some examples of terms in a language talking about the real numbers include 0, sum(0, 1), multiply(sum(0, 1), sum(1, 1)) etc.
A formula on the other hand is a statement about the terms. It says something about them. In standard logic, any given formula would be either true or false. Note that some texts may specifically required that formulas be referred to as well-formed formulas or WFFs. In logic, we generally tend to define things recursively. To give an example of how recursive definitions work, we define a formula and term here recursively.

Definition 2.3 (Terms in a Language)**.** A term in a language is defined as follows:

(1) A variable is a term.
(2) A constant symbol is a term.
(3) If F is a m-placed function symbol, and $t_1, ..., t_m$ are terms, then $F(t_1, ..., t_m)$ is a term.
(4) A string of symbols is a term if and only if it can be shown to be a term through a finite series of applications of (1), (2), and (3).

Definition 2.4 (Formula in a language)**.** A formula in a language is defined as follows:

(1) If t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.
(2) If t_1, t_2,...,t_n are terms and R is a n-placed relation symbol, then (R(t_1, t_2,...,t_n)) is a formula.
(3) If ψ is a formula, then $\neg\psi$ is also a formula.
(4) If ψ and ϕ are formulas, then so are $\psi \wedge \phi$, $\psi \vee \phi$, $\psi \implies \phi$, and $\psi \iff \phi$.
(5) If ψ is a variable and v_i is a variable, then $(\forall v_i)\psi$ and $(\exists v_i)\psi$ are formulas.

(6) Any string of symbols in our language is only a formula if it can be shown to be a formula starting from some terms and applying rules (1), (2), (3), (4), and (5) finitely many times.

Any component of a formula that could be a formula in itself is called a subformula. We don't define a subformula here formally, but here is an example: Subformulas of a formula $(\forall v_i(R(v_i) \implies \psi) \vee (\neg\phi)$ include $\forall v_i(R(v_i)$, ψ, ϕ etc.

Definition 2.5 (Bound and free variables)**.** A variable v_i is said to occur bound in a formula if ψ if and only if for some subformula ϕ of ψ either $(\forall v_i)\phi$ or $(\exists v_i)\phi$ is a subformula of ψ. If a variable isn't bound in a formula, it's said to be free.

A variable can occur both bound and free in the same formula. For example, in the formula $(\forall v_0)(v_0 = v_1) \wedge F(v_0)$, where F is a 1-placed function symbol, the variable v_0 occurs both bound and free.

Definition 2.6 (Sentence)**.** A sentence is a formula in which no variable occurs freely.

3. Models and Theories

A model $\mathfrak{U}$ of a language $\mathcal{L}$ is like an 'instance' of a language. For example, a language $\mathcal{L} = 0,\ 1,\ +,\ \cdot,\ \leq$ could have multiple models- the real numbers, the rational numbers, the integers etc, as each of them have the same functions, relations and constants as $\mathcal{L}$. As we can see, what model of the language we're using somehow depends on a different set $\mathbf{A}$, one to which our constants belong, and such that our functions and relations take elements of $\mathbf{A}$ as input.

Definition 3.1 (Model of a Language)**.** A model (or structure) $\mathfrak{U}$ of a language can be defined as an ordered pair $((\text{A}), \mathcal{I})$ where $\mathcal{I}$ is an **interpretation function**. This interpretation function maps the elements of our language to concrete, well-defined mathematical objects we're working with. We can formally define this as follows:

(1) If c is a constant symbol in our language, then $\mathcal{I}(c) \in (\text{A})$. Here $\mathcal{I}(c)$ is called a constant.
(2) If F is a m-placed function symbol, then $\mathcal{I}(F)$ is a m-placed function symbol from $\mathbf{A}^m$ to $\mathbf{A}$
(3) If R is a n-placed relation symbol, then $\mathcal{I}(R)$ is a n-placed relation symbol, such that for all $a_1, a_2, .., a_n \in \mathbf{A}$, $\mathcal{I}(R)(a_1, a_2, .., a_n)$ is a formula.

In the context of a model, a term stands as an element in **A**, the value of which is calculated using some rules that depend on the specific mathematical object we're working with.

The primary idea of model theory is that we can think of mathematical objects as models of a language.

For every term t, $\mathcal{I}(t)$ is a term that belongs to **A**. The value $t(x_0, ..., x_n)$ of a term $t(v_0, ..., v_n)$ is the universe **A** for a model $\mathfrak{U}$ is defined the way we would expect it to, similar to how polynomials are calculated in algebra.

Definition 3.2 (Satisfaction of a Formula)**.** Let $\mathfrak{U}$ be a model of a language $\mathcal{L}$. The sequence $(x_0, x_1, ..., x_n)$ of variables is said to satisfy a formula $\mathfrak{U} \vDash \psi(v_1, v_2, ..., v_n)$, written as $\psi(v_1, v_2, ..., v_n)$, if the following hold-

(1) If ψ is the formula $t_1 = t_2$, then the interpretations of t_1 and t_2 in the model $\mathfrak{U}$ are equal, that is $t_1[x_1, ..., x_n] = t_2[x_1, ...x_n]$
(2) If ψ is the formula $R[t_1, ..., t_n]$ for a n-placed relation R, then $x_1, .., x_n$ satisfy ϕ if and only if $\mathfrak{U} \vDash S(t_1[x_1, ...x_n], ...t_n[x_1, ...x_n])$
(3) if ψ is the formula $\neg\phi$, $\mathfrak{U} \vDash \psi$ iff $\mathfrak{U} \nvDash \phi$.
(4) if ψ is the formula $\phi \wedge \theta$, $\mathfrak{U} \vDash \psi$ if and only if $\mathfrak{U} \vDash \phi$ and $\mathfrak{U} \vDash \theta$. Formula satisfaction for other connectives is defined in a similar manner.
(5) If ψ is the formula $(\forall v_i)\phi$, then $\mathfrak{U} \vDash \psi$ iff for every x in **A**, $\psi \vDash \phi[x_0, .., x_i - 1, x, x_i + 1, .., x_n]$ The case for $\exists$ works similarly.

Note that that the notation $\mathfrak{U} \vDash \psi$ is used if ψ is satisfied for every assignment from **A**.

Definition 3.3. If Σ is a set of sentences, $\mathfrak{U}$ is called a **model** of Σ, written as $\mathfrak{U} \vDash \Sigma$, if for every sentence $\sigma \in \Sigma$, $\mathfrak{U} \vDash \sigma$.

Definition 3.4. A set of sentences Σ is said to be **satisfiable** if there exists a model $\mathfrak{U}$ of Σ.

Definition 3.5. A **theory** $\mathcal{T}$ is a set of sentences. If $\mathcal{T}$ is a theory and σ is a sentence, $T \vDash \sigma$ if for all models $\mathfrak{U}$, if $\mathfrak{U} \vDash T$ then $\mathfrak{U} \vDash \sigma$.

Definition 3.6. For a model $\mathfrak{U}$ of a language $\mathcal{L}$, the **theory of** $\mathfrak{U}$ is the set of all sentences of $\mathcal{L}$ which are true in $\mathfrak{U}$, that is the set of all sentences σ so that $\mathfrak{U} \vDash \sigma$.

Definition 3.7. $\Sigma \in \mathcal{T}$ for a theory T is said to be the **axioms** for a theory $\mathcal{T}$ if $\Sigma \vDash \sigma \forall \sigma \in \mathfrak{T}$.

4. Godel's Completeness Theorem

4.1. **Semantic and Syntactical Implication.** So far, the idea of elements of a set A satisfying a formula ψ in a language $\mathfrak{L}$ that we've seen has been based on the idea that we interpret the statement in our model. This interpretation depends on what kind of mathematical objects we're working with. For a simple example, if we have a language containing a 2-placed relation symbol R and we have a model of the real numbers, we could interpret R as either a lesser-than or the greater-than symbol, and the statement S(0, 1), where S is $\mathfrak{I}$(R) will be true and false in the two different cases. This idea of a statement being true, which depends on how we're interpreting the formula in our model is known as semantic entailment. We use the symbol $\vDash$.

On the other hand, there's a purely mechanical way of deriving true statements from true statements based on the idea of a deductive calculus. This deductive calculus is based on the idea of deduction rules, that allow us to derive true statements from true statements. If a set of sentences Γ syntactically implies σ if we can show that using the deduction rules. We write this as $\Gamma \vdash \sigma$. For a basic example, we can conclude that any formula $\psi = \phi \wedge \theta$ syntactically implies both ϕ and θ, as in ϕ and θ are both always true if ψ is true.

4.2. **Model Existence Theorem.** This theorem is known as the model existence theorem and is equivalent to Godel's completeness theorem for first order logic, which we will soon get to.

Definition 4.1. We say that a set of formulas Γ **decides** a formula ψ if either $\Gamma \vdash \psi$ or $\Gamma \vdash \neg\psi$. This means that we can prove that ψ is either true or false from Γ is the standard sense of proving something.

Considering that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$ are formulas in our language, neither of them may be provably true using our methods (which we have in the form of something called introduction and elimination rules). However, for a given assignment of variables from our set **A**, we always have a defined truth value.

Definition 4.2. A set of sentences Σ is said to be consistent if there is no formula ψ such that $\Sigma \vdash \psi$ and $\Sigma \vdash \neg\psi$.

Theorem 4.1 (Model Existence Theorem). Every set of consistent sentences has a model.

We don't prove the model existence theorem here, as it's complicated and requires a solid background in mathematical logic. In general, one way to do the proof is by Henkin's construction. We show that every theory can be extended to one that is maximally consistent (contains the maximum number of sentences possible while remaining consistent), and containing something called the witness property. It is then quite easy to show that a theory that is maximally consistent and has the witness property has a model. It can be then be shown that this model is also a model of the original theory.

4.3. **Godel's Completeness Theorem.**

Theorem 4.2 (Godel's Completeness Theorem)*.* If $\Sigma \vDash \phi$ then $\Sigma \vdash \phi$

Proof- Assume that $\Sigma \vDash \phi$ Then, $\mathfrak{U} \vDash \Sigma \to \mathfrak{U} \vDash \phi$ for all structures $\mathfrak{U}$ (by definition of $\Sigma \vDash \phi$ Therefore, $\Sigma \cup (\neg phi)$ has no models. By the contrapositive of the model existence theorem (if a set of sentences doesn't have a model it's inconsistent), $\Sigma \cup \neg\phi \vdash \bot$. By the Deduction Theorem in logic, it follows that $\Sigma \vdash (\neg\phi \to \bot)$. From here, we get $\Sigma \vdash \phi$ from the tautology (something that's always true) $(\neg\psi \to False) \to \psi)$

5. Compactness Theorem and it's Applications

One direction is trivial: if $\mathfrak{U} \vDash \Sigma$, then $\mathfrak{U} \vDash \Sigma_0$ for every finite $\Sigma_0 \in \Sigma$.

For the other direction (Σ has a model if every finite subset has a model), we show that if Σ doesn't have a model, there exists some finite $\Sigma_0 \in \Sigma$ such that Σ_0 doesn't have a model. Since Σ doesn't have a model, $\Sigma \vDash \bot$. By the completeness theorem, it follows that $\Sigma \vdash \bot$. This means that there exists a deduction of a contradiction from Σ through a series of deductive sentences $(\delta_1, \delta_2, ..., \delta_n)$. Let Σ be defined as $\Sigma \cap (\delta_1, \delta_2, ..., \delta_n)$. Σ_0 is a finite set. Also, $\Sigma_0 \vdash \bot$, as the same sequence $(\delta_1, \delta_2, ..., \delta_n)$ is the deduction of of $\bot$ from Σ_0.

Therefore $\Sigma_0 \vDash \bot$ by the soundness theorem (the opposite direction of the completeness theorem, stating that if $\Sigma \vdash \psi$ the $\Sigma \vDash \psi$). This means that Σ_0 has no models.

This completes the proof of the compactness theorem.

5.1. **Non-Standard Models of Number Theory.**

A non-standard model of Peano arithmetic is a model of first-order Peano arithmetic that includes non-standard numbers (numbers that are greater than all natural numbers).

The language of natural numbers is traditionally (N, +, $\cdot$, 0, 1). We add a constant c to the language $\mathcal{L}$ to get (N, +, $\cdot$, 0, 1) $\cup$ c. Along with the standard axioms of Peano arithmetic, we include the infinite set of axioms c ¿ n for each natural number n. We know that this theory has models by the compactness theorem.

Any finite subset of these axioms is satisfied by a model that is the standard model of arithmetic plus the constant x interpreted as some number larger than any numeral in the finite subset.

5.2. **Model Theory for Translating Between the Finite and Infinite.**

Theorem 5.1. If a theory $\mathcal{T}$ has arbitrarily large finite models, it has an infinite model.

Proof. Consider new constant symbols c_i for $i \in \mathbb{N}$, the natural numbers and expand from $\mathcal{L}$, the language of $\mathcal{T}$ to $\mathcal{L}' = \mathcal{L} \cup c_i : i \in \mathbb{N}$.

$\Sigma = T \cup \neg c_i = c_j : i \neq j, i, j \in N$ Every finite subset of Σ has a model if we interpret the finitely many constant symbols as different elements in an expansion of finite model of $\mathcal{T}$. By compactness we have a model $\mathfrak{U}$' of Σ. □

This is an extremely important theorem with many applications.

(1) **The four-color theorem:** Any planar graph can be four-colored. For finite graphs, this is the famous result of Appel and Haken. Model theory takes us from the finite to the infinite. A plan graph is one that can be drawn in Euclidean plane and to be four-colored means that that each vertex of the graph can be assigned one of 4 colours such that no connected pair of vertices has the same color. We consider $\mathfrak{U}$ to be a finite planar graph with four unary relations symbols: R, G, B and Y (for red, green, blue and yellow). Model theory allows us to show that there is some expansion $\mathfrak{U}$' of $\mathfrak{U}$ such that $\mathfrak{U}$' $\models \sigma$ whenever σ is the sentence in the expanded language: $(\forall x)[R(s) \vee G(x) \vee B(x) \vee Y(x)] \wedge (\forall x)[R(x) \implies \neg(G(x) \vee B(x) \vee Y(x))] \wedge ...$ This will ensure that the interpretation of R, G, B and Y will color the graph.

(2) Model Theory can also allow us to go from the infinite to the finite. **Ramsey's Theorem:** For each n $\in \mathbb{N}$, there is an r $\in \mathbb{N}$ such that if $\mathfrak{G}$ is any graph with r vertices, then either $\mathfrak{G}$ contains a complete subgraph with n vertices or a discrete subgraph with n vertices. Ramsey began by proving an infinite version of the theorem, which can then be adapted for finite graphs using model theory.

(3) **Konig's Infinity Lemma**- Every infinite tree contains a vertex of infinite degree or an infinite simple path.

6. Other Applications

6.1. **Infinitesimals.** We state here without proof the Leibniz Principle, the shortest proof of which is using model theory. Theorem- There is an ordered field called the hyperreals, containing the reals R and a number larger than any real number such that any statement about the reals which holds in R also holds in *R.

The element b $\in *R$ which is larger than all real numbers gives rise to an inifnitesimal $1/b \in *R$. An element x is called an infinitesimal if $-1/n < x < 1/n foreach n \in N$. Zero is an infinitesimal.

Infinitesimals provide an alternate approach for doing calculus in contrast to the epsilon-delta limits based approach that is mostly used today. This alternate approach is known as nonstandard and is close in spirit to how the original developers of calculus intended it to look like. Using infinitesimals also makes the proofs of many theorems easy.

6.2. **Hilbert's Seventeenth Problem.** By assuming that the theory of RCF (real closed fields, like the reals) holds, we can provide a solution to **Hilbert's 17th problem** from his famous list of 23 problems for the 20th century. The problem asks if given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions.

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