Analytic Combinatorics - Singularity Analysis

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Motivation

The idea is as follows: a common question in combinatorics is "How many objects of a given size are there in a combinatorial class?" In analytic combinatorics, when we cannot answer this, we ask, "Approximately how many are there? Can we bound the error? If not, how does the error grow?"

Preliminaries: Generating Functions and Asymptotics

Asymptotics:

- $f(x) \sim g(x) \leftrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ (asymptotic equivalence)
- $\lim_{x\to\infty} \frac{O(f(x))}{f(x)} = c$ for a constant c that is not ∞ (Big O Notation)
- $\lim_{x\to\infty} \frac{o(f(x))}{f(x)} = 0$ (Little o notation)

Generating Functions: Given a sequence a_n for non-negative integral n, we can encode this sequence into a generating function, f, which satisfies

- (OGF) $f(x) = \sum_{n=0}^{\infty} x^n \cdot a_n$,
- (EGF) $f(x) = \sum_{n=0}^{\infty} \frac{x^n \cdot a_n}{n!}$,

Stirling's Formula:
$n! \sim \sqrt{2\pi n} \cdot (\frac{n}{e})^n$.

Preliminaries: Complex Analysis Part 1

In the next 3 slides, we will discuss various aspects of complex analysis related to analytic combinatorics.

- Think of a complex function $f(z)$ as a transformation of space.
- $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ (h is complex).
- (Cauchy-Reimann Equations) If f is holomorphic, and $f(x+yi) = p(x,y) + iq(x,y)$, then

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \textit{ and } \frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y}.$$

- f is holomorphic iff it is analytic, meaning that it can be expressed as a power series that locally converges to the function.
- (Cauchy-Hadamard's Theorem) A power series converges in the interior of a circle with radius $1/\limsup_{n\to\infty} |a_n|^{1/n}$.

Preliminaries: Complex Analysis Part 2

- The analytic continuation of a function $f(z)$ over a domain where f is undefinied is the *unique* function $g(z)$ that is holomorphic on this domain and f's and equal f when their domains intersect.
- A singularity is a point that must be excluded from any analytic continuation. A pole is a singularity (locally) of the form $1/(z-a)^n$ for positive integral n.
- For some singularities, for example, of the form $\sqrt{z-a}$, a branch cut is necessary.
- There must be at least 1 singularity on the boundary of convergence of a function's power series.
- (Vivanti-Pringsheim Theorem) If the power series of f has non-negative coefficients, the closest (or one of the closest) singularity is on the positive half line.
- (Laurent Series) Many complex functions can be represented by a power series plus a sum of the form $\sum_{n=1}^{m} \frac{b_n}{(z-a)^n}$.

Preliminaries: Complex Analysis Part 3 - Contour Integration + Gamma Function

- A contour integral is of the form $\int_\gamma f(z)dz$, where $\gamma(t) \in \mathbb{C}$, $t \in [a, b]$.
- Cauchy's Coefficient Formula: $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} dz$ where γ is a closed contour inside f's domain where it is analytic.
- The Gamma Function, defined as $\Gamma(z) = \int_0^\infty t^{(z-1)} e^{-t} dt$, is a contiuous representation of $(n-1)!$, and though converges for $Re(z) > 0$, can be analytically continued to the entire complex domain, except for poles at 0 and the negative integers.

Exponential Estimates

By the Cauchy-Hadamard Theorem, the location of singularities (specifically the dominant, or closest ones) provides important information about the asymptotic scale of coefficients.

If r is the distance from the origin to the nearest singularity, then

$$a_n = (1/r)^n \cdot \theta(n),$$

where $\theta(n)$ is a sub-exponential function satisfying $\limsup_{n\to\infty} |\theta(n)|^{1/n} = 1$

Meromorphic Aymptotics Approach

Given a generating function $f(z)$, which has no singularities in the complex plane on the circle centered at the origin with radius r, and only pole singularities within this circle, assume that there are j poles inside, and the kth pole is located at p_k. Assume that the pole is of multiplicity β_k, and the Laurent series expansion of the f at p_k is $f(z) = F(z) + \sum_{m=1}^{\beta}(\frac{c_{k,m}}{(z-p_k)^m})$, where F is analytic. Then the coefficient of z^n in the Maclaurin expansion of f, a_n, satisfies

$$\boxed{a_n \sim \sum_{k=1}^{j} \sum_{m=1}^{\beta_k} \frac{c_{k,m}}{(-1)^m (p_k)^{m+n}} \cdot \binom{n+m-1}{n}} = h(n)$$

(this is an exponential-polynomial on n) with error $\epsilon_n = |h(n) - a_n| \leq \frac{M}{r^n}$ where $M = \sup_{z \in \gamma} |f(z)|$, where γ is the contour of the counterclockwise oriented circle with radius r surrounding the origin.

Example - Zag Numbers Part 1

For odd n, consider the number of permutations of size n such that each element is alternatingly greater than or less than the previous (let this be a_n) (we start with an increase (up-down)). For example, if n $= 5$, then 2, 4, 1, 5, 3 is an alternating permutation. One can see that the number of down-up alt. permutations is equivalent to up-down, by subtracting each element from $n+1$, so we can get the recurrence $2a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k}$ by iterating on the location of the largest element. If $f(x)$ is the EGF, $2\frac{df}{dx} = (f(x))^2 + 1$ (the $+1$ because $2 * a_1 = 2 = 1 + a_0^2 = 2$) Solving this differential equation by separation of variables with I.C. $a_0 = 1$, we get $f(x) = tan(x/2 + \pi/4) = tanx + secx$. Since n is odd, we care about the coefficients of the Maclaurin expansion of $tanx$ multiplied by $n!$.

Example - Zag Numbers Part 2

Notice that $tan(a+bi)$ only has singularities when $cos(a+bi) = cos(a)cos(bi) - sin(a)sin(bi) = (e^b + e^{-b})cos(a)/2 - i(e^b - e^{-b})sin(a)/2 = 0$, which only occurs at $a + bi = \pi/2 + k\pi$ for any $k \in \mathbb{Z}$. These singularities are poles, since *sin* and *cos* are entire (analytic on all $\mathbb{C}$), so meremorphic asymptotics applies. $sin(\pi/2) = 1$, and $cos(x) = cos((x-\pi/2)+\pi/2) = -sin(x-\pi/2) = -(x-\pi/2)(1+(x-\pi/2)^2 F(x-\pi/2))$, where F is an analytic function at 0. Therefore, $tanx = \frac{-1}{x-\pi/2} + G_1(x-\pi/2)$ for some analytic G_1 at 0, and similarly $tanx = \frac{-1}{x+\pi/2} + G_2(x+\pi/2)$ for analytic G_2. The rational approximation for $tanx$ at these singularities is $\frac{2/\pi}{1-x\cdot 2/\pi} - \frac{2/\pi}{1+x\cdot 2/\pi}$.
Therefore, $\frac{a_n}{n!} \sim 2 \cdot (\frac{2}{\pi})^{n+1}$ for odd n, or by Stirling's Formula, $\boxed{a_n \sim \frac{4\sqrt{2\pi n}}{\pi} \cdot (\frac{2n}{\pi e})^n}$, with a possible error bound $\frac{M}{\pi^n}$, where $M = \sup_\gamma tan(z)$ for γ being the circle centered at the origin with radius π.

Singularity Analysis Theorems Page 1

Singularity Analysis is a more robust extension of Meromorphic Asymptotics which can analyze various types of singularities that are not poles. Here, we present the fundamental theorems and will discuss the proofs. We denote the nth coefficient of $f(z)$ as $[z^n]f(z)$. Note that we assume singularities are at 1, because, based on the exponential factor theory reviewed in slide 7, we can analyze the location and nature of a singularity seperately.

- $[z^n](1-z)^{-\alpha} = \frac{\prod_{k=0}^{n-1}(\alpha+k)}{n!} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \cdot (1 + \sum_{k=1}^{j-1} \frac{e_k}{n^K}), \alpha \in \mathbb{C}, e_k = \sum_{l=k}^{2k} \gamma_{k,l} \prod_{m=1}^{l}(\alpha - m), \gamma_{k,l} := [v^k t^l]e^{-t}(1-vt)^{-1-\frac{1}{v}}$ with error $O(\frac{1}{n^j})$.
- $[z^n](1-z)^{-\alpha}(\frac{1}{z} ln(\frac{1}{1-z}))^{\beta} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(ln(n))^{\beta} \cdot (1 + \sum_{k=1}^{j-1} \frac{C_k}{(ln(n))^k}) = h(n), C_k = \frac{\prod_{m=0}^{k-1}(\beta-m)}{k!}\Gamma(\alpha) \cdot (\frac{d^k}{ds^k}(\frac{1}{\gamma(s)}))|_{s=\alpha}$ with error $O(1/(ln(n))^j)$. If α is a negative integer k or 0, consider the limit as $\alpha \to k$ (or 0), to see only a slight alteration in the formula. Also, one can derive better error estimate of the scale $O(1/n^m)$ if β is a positive integer, which is relatively simple by following a similar process to the proof of the original asymptotic, but we will not go into details now.

Singularity Analysis Theorems Page 2

- (HLK Tauberian Theorem) If A(n) satisfies $\lim_{n\to\infty} A(cn)/A(n) = 1$, for $c > 0$, then A is slowly varying, and $[z^n](1-z)^{-\alpha}A(\frac{1}{1-x}) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}A(n)$.
- If $f(z) = O((1-z)^{-\alpha}(ln(\frac{1}{1-z}))^{\beta})$, then $[z^n]f(z) = O(n^{\alpha-1}(ln(n))^{\beta})$, and similarly, if $f(z) = o((1-z)^{-\alpha}(ln(\frac{1}{1-z}))^{\beta})$, then $[z^n]f(z) = o(n^{\alpha-1}(ln(n))^{\beta})$. A similar result holds for the HLK Tauberian Theorem.
- If $f(z) \sim (1-z)^{-\alpha} \cdot (\frac{1}{z}ln(\frac{1}{1-z}))^{\beta}$, then $[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(ln(n))^{\beta}$ and a similar result holds for the HLK Tauberian Theorem (this collorary follows directly from the previous bullet and the fact that $f(x)\ g(x) \leftrightarrow f(x) = g(x) + o(g(x))$).

Example Part 1

Problem: Define a Dyck-extension object to be an object of size n which is composed of a Dyck word (a permutation of an equal number of As and Bs where at no point there are more Bs than As) of length $2n-2$ and a permutation of the integers from 1 through n. Find an asymptotic approximation for the number of sets of Dyck-extension objects that have a sum of lengths of n. **Solution:** Notice that regular Dyck words are all of the form A"dyck word"B"dyck word," giving Segner's reccurence $c_{n+1} = \sum_{k=0}^{n} c_{k-n}c_k, c_0 = 1$ for c_n as the number of Dyck words of length 2n. The OGF of c_n, satsfies $c(x) = 1 + x(c(x))^2$, and therefore $c(z) = \frac{1-\sqrt{1-4x}}{2x}$, but since we care about the number of Dyck words of length $2n-2$, our generating function is $\frac{1-\sqrt{1-4x}}{2}$. Since we are adding a permutation of n integers to each object, we are multiplying each coefficient by $n!$, so it suffices to treat our function as an EGF. We have to compose our EGF with e^x, as we are considering a set of Dyck-extension objects, so we end up with the EGF $e^{\frac{1-\sqrt{1-4z}}{2}} = f(z)$.

Example Part 2

Now, we begin the analytic combinatorics stage. $f(z) = \sqrt{e} \cdot (1 - \frac{\sqrt{1-4z}}{2} + \frac{1-4x}{8} - \ldots) = -\sqrt{e} \cdot \frac{\sqrt{1-4z}}{2} + F_1(z) + (1-4z)^{3/2} \cdot F_2(z)$, where F_1, F_2 are analytic at the singularity $z = 1/4$. Therefore, we have that the desired coefficient is $\sim 4^n \cdot \frac{-\sqrt{e}}{2} \cdot \frac{1}{n^{1.5}\Gamma(-0.5)} = 4^{n-1}\sqrt{\frac{e}{\pi}} \cdot n^{-1.5}$ with error $O(n^{-2.5})$. Using Stirling's formula, we get that the desired quantity is $\sim 4^{n-1}\sqrt{e} \cdot n^{-1.5} \cdot \sqrt{2n} \cdot (n/e)^n$.

Overview of Saddle Point Method Approach

The saddle point method is an alternate approach to determining asymptotics, which greatly differs from singularity analysis.

The idea is as follows: By Cauchy's Coefficient Formula, $[z^n]f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz$, which, by substitution, can be expressed as an integral with real bounds and complex integrand. Then, we can write the integrand as $e^{F(z)}$ for some function F, and after determining the maximum (called the saddle point because, in the complex plane, the only stationary points are saddle points (can be proven by considering directional derivatives of the function mapping complex value to magnitude of the function)), F, can locally be approximated by a constant, a squared term, and a function on the scale of $O((n-a)^3)$. This allows us to approximate the integral as a gaussian integral, provided that we can complete the tails of the gaussian integral (1), ignore the tails of the original integral (2), and that the cubic term can be ignored. We choose the main part of the contour (not the removed tails) to be large enough such that (1, 2) are satisfied (a heruistic is $\lim_{n\to\infty} l^2 \cdot f''(a) = +\infty$, where a is the saddle point and l is the main contour length), but small enough such that (3) is satisfied ($\lim_{n\to\infty} l^3 \cdot f'''(a) = 0$).

Outline of an Example

We will determine an asymptotic for $[z^n]e^z$, which, when taking the reciprocal, proves Stirling's formula. $[z^n]e^z = [z^n]f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz$ (γ is a counter-clockwise oriented circle with radius r,) which, by substitution, $= \frac{e^n}{2\pi n^n} \cdot \int_{-\pi}^{\pi} e^{n(e^{i\theta}-1-i\theta)}d\theta$ (we set $r = n$, because, in this integral, it places the saddle point (where the first derivative of the integrand is 0) on the real line (at $\theta = 0$)). We shift the interval of integration and split it into the dominant part, on the interval $[-\theta_0, \theta_0]$, and the part that tends to 0 $\theta_0, 2\pi - \theta_0$. In order to satisfy the heruistic mentioned in the previous slide, we set $\theta_0 = n^{-2/5}$ (n^α would work for $\alpha \in [-\frac{1}{2}, -\frac{1}{3}]$). By Taylor approximation of $n(e^{i\theta} - 1 - i\theta)$, and u-substitution, we can transform the integral into an approximate Gaussian integral, which leads us to the asymptotic $[z^n]e^z = \frac{1}{n!} \sim \frac{e^n}{n^n\sqrt{2\pi n}}$. We can then rigorously show that the other integral approaches 0 as $n \to \infty$, that the tails of the Gaussian Integral tend to 0 (so we can add them), and that the non-quadratic part of the Taylor Expansion approaches 0 when considered in the integral.

End

Thanks for your attention!