ANALYTIC COMBINATORICS - SINGULARITY ANALYSIS

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ABSTRACT. In this expository paper, we review the elegant field of Analytic Combinatorics, with a particular focus on Singularity Analysis. We cover the topics of Meremorphic Asymptotics, Singularity Analysis, the Saddle Point Method, and touch on Multivariate Asymptotics (ACSV). We go over the necessary theorems with proofs, and provide practice examples to help illustrate the approaches. We provide all necessary background in Complex Analysis, which is an essential subject for Analytic Combinatorics, in the preliminaries section, so we only require an understanding of single variable calculus and a few topics from multivariate calculus. We hope to, with this paper, equip the reader with the necessary tools in Analytic Combinatorics to begin exploring topics such as ACSV in more detail and read/understand research articles in the field.

1. Introduction

Analytic combinatorics is a beautiful field that tries to approach problems in a different and elegant way. The main goal of combinatorics is often to find the number of objects with a given size in a combinatorial class. Oftentimes, it is very difficult, if not impossible to answer this question.

In these cases, we can rely on the field of Analytic Combinatorics, which asks the following questions: can we approximate the number of objects? If so, can we bound the error? If we cannot bound the error, can we examine how the error grows? Can we improve our asymptotic estimates by decreasing the error?

Analytic Combinatorics is generally useful when no known formula to compute a desired quantity can be computed in $O(1)$ time. In these cases, we may determine *asymptotics* that are computable in $O(1)$ time. These *asymptotics* are generally determined by analyzing the *singularities* of *generating functions* for these quantities with tools from *complex analysis*. We will explain all the terminology utilized in this snapshot in our preliminaries section.

Much of this paper includes information from Analytic Combinatorics, by Flajolet and Sedgewick, which is generally considered the gold-standard of the subject. In this paper, we hope to summarize the key ideas in about 20 pages in a manner that is very intuitive for the reader, and also include information from outside sources to further illustrate our points, hopefully providing a comprehensive and detailed view of the subject of Analytic Combinatorics, with a focus on the field of Singularity Analysis. We also go into depth about Complex Analysis preliminaries so that any reader who is unfamiliar with the subject will be able to understand the material.

There are numerous applications of Analytic Combinatorics, from various aspects of combinatorics (especially graph theory) to probability laws (multivariate asymptotics), and even outside of mathematics (to areas such as chemistry and bio-chemistry; for example, the number of isomers of alcohols can be analyzed utilizing Analytic Combinatorics (Polya's Alcohols)). It is out of the scope of this paper to go into depth about all the applications,

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but we mention some of them in the practice problems, and we encourage the reader, if interested, to learn more about applications.

2. Preliminaries

2.1. **Prior Knowledge.** For this expository paper, we hope to build all of the major theorems and methods based on as little preliminary knowledge as possible, but we do require an understanding of real single variable caluclus, as it is unfortunately not possible to reprove all of the necessary preliminary theorems in the scope of this approximately 20 page paper. We will also be utilizing some aspects of multivariable calculus, but these aspects are sparse enough that, if the reader has not taken a multivariate calculus class, a few online searches should be sufficient. It also helps, though not necessary, to be familiar with asymptotic equivalence, complex analysis and the usage of generating functions in combinatorics (if so, these sections may be skipped), but we will outline all of the aspects of these topics that we will utilize in this paper below.

2.2. **Asymptotics and Error.** A basic understanding of the goal of analytic combinatorics is to determine asymptotics for combinatorical quantities via generating functions and complex analytic methods. Usually, for these asymptotics, we also would like to include information about the error. Evidently, we must first understand, from a mathematical standpoint, what we mean by "asymptotics" and "error."

We denote asymptotic equivalence by the symbol $\sim$, for example,

$$f(x) \sim g(x).$$

This is essentially shorthand for saying

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

and is a way of comparing/specifying the growth of functions. Usually, one of these functions is a quantity that we want to study and that has a non-constant computational complexity (which we will elaborate on this later) to compute, while the other is an algebraically simplified function that can quickly be computed in $O(1)$ time. For example, the famous prime number theorem states that $\pi(x) \sim \frac{x}{\ln(x)}$, where $\pi(x)$ is the number of primes less than or equal to x. For asymptotic equivalence, x is generally $\in \mathbb{Z}$, $\in \mathbb{R}$, or $\mathbb{C}$. Also, sometimes (rarely), x is explicitly stated to approach a different value other than ∞. For instance, $f(x) \sim g(x)$ for $x \to a$ for some a, for example should be read as

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

Also, the notation $g(x) = O(f(x))$ and $g(x) = o(f(x))$ represent mathematically that $\exists c, s \ni \forall x > c \ni g(x) \leq s \cdot f(x)$ and $\exists c, s \ni \forall x > c \ni g(x) < s \cdot f(x)$, respectively ($\lim_{n\to\infty} O(f(n))/f(n)$ is zero or some other constant (not infinity), and $\lim_{n\to\infty} o(f(n))/f(n) = 0$), or intuitively represent an asymptotic notion of less than or equal to and less than, respectively. These notations are often substituted into formula to represent parts that grow at most at a given asymptotic rate or less than a given asymptotic rate, respectively. Again, similar to the asymptotic equivalence $\sim$, we generally assume that $x \to \infty$ (which yields the above formulas), but we can specify x to approach some value a, therefore the formulas changing to $g(x) = O(f(x)) \leftrightarrow \exists \epsilon, s \ni \forall 0 < |x-a| < \epsilon \ni g(x) \leq s \cdot f(x)$ and $g(x) = o(f(x)) \leftrightarrow \exists \epsilon, s \ni \forall 0 < |x-a| < \epsilon \ni g(x) < s \cdot f(x)$ One major distinction between asymptotic equivalence and the O and o notations is that, in the former case, the scaling coefficient, is known, while it is unknown in the latter. Big O notation (and small o) are often used to express the growth of the number of basic operations necessary for a program to output a result (called the computational complexity), but in the case of analytic combinatorics, we often use it to understand error bounds on asymptotics for the combinatorical quantity of interest. One important relation is that,

$$f(x) \sim g(x) \leftrightarrow f(x) = g(x) + o(g(x)),$$

no matter whether $x \to \infty$ or $x \to a$ for some value a (this will show up later in our chapter on singularity analysis). Lastly, it is useful to understand our specific goals in analytic combinatorics. As in most branches of combinatorics, the prefered scenario is where we have a nice formula to compute the combinatorical quantity of interest in $O(1)$ time. If this is not possible, we usually want a way of computing the exact quantity in a relatively low computational complexity, as well as an asymptotic equivalence for the quantity that can be computed in $O(1)$ time (the latter is the case in which analytic combinatorics is usually necessary). In this case, we would ideally want some understanding of the error of our asymptotic expansion. The best case is where we have an explicit formula that bounds the error in a relatively close manner. If this is not possible, then the next best scenario is when we have the error in terms of a big O or small o notation. Sometimes, even this is not possible, and we are only able to represent the asymptotic equivalence of the quantity without any notion about the error (except, of course, that the error grows slower (small o notation) than the asymptotic itself, though this is trivial).

2.3. **Generating Functions.** It is central to the discussion about analytic combinatorics to understand generating functions, as the method works by determining asymptotics for the coefficients of power series (as we will see, this is applicable in the case of OGFs, EGFs, and MGFs). Consider a sequence a_n for integral $n \geq 0$. We can "encode" this sequence into a function in a number of ways. The OGF (Ordinary Generating Function) is $f(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n$ for some center of the power series c (usually 0), in power series representation, though we generally want to have an nice, algebraically simplified representation of $f(x)$ (or sometimes, a functional equation also is useful). Similarly, the EGF (Exponential Generating Function) is $f(x) = \sum_{n=0}^{\infty} a_n \cdot \frac{(x-c)^n}{n!}$, which is the Taylor Series representation of f(x) around c (the exponential generating function coefficients can be easily turned into the coefficients for the ordinary generating function by multiplying by $n!$). The DGF (Dirichlet Generating Function) representation is slightly unique, and is defined as $f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^x}$. This representation is especially useful for a number theory perspective, for reasons that will become apparent below (hint for the interested reader: consider what happens when one multiplies 2 DGFs together). A Multivariate Generating Function (MGF) is a higher dimensional analog of generating functions for sequences based on multiple integral indexes, for example, $a_{l,m,n}$, for non-negative integral l, m, n, for which the generating function is $f(x, y, z) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{l,m,n} \cdot (x-x_0)^l (y-y_0)^m (z-z_0)^n$, for some x_0, y_0, z_0 that are usually (unless otherwise specified) all 0, and similarly for higher dimensions. For the scope of this paper, we will focus on OGFs and EGFs unless another type of generating function is explicitly stated. Also, when we say "coefficient n", we mean the value a_n corresponding to $f(x)$. As mentioned above, generating functions are closely related with Taylor's series, and have many useful properties that will be listed below (based on the definitions above (and if multiple generating functions are necessary, we will use $f(x)$, $g(x)$ and $h(x)$), corresponding to the sequences a_n, b_n, c_n):

- For OGFs, $f^{(n)}(c)/(n!) = a_n$, and for EGFs, $f^{(n)}(c) = a_n$.
- (Cauchy Coefficient Formula - Will be elaborated on later) For OGFs, $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}} \cdot dz$, where γ is a contour in the complex plane circling the origin counter-clockwise and is in a domain in which f is analytic. EGF coefficients can be similarly calculated after multiplying by $n!$.
- For OGFs, the nth coefficient of $f(x)g(x)$ is $\sum_{k=0}^{n} a_k \cdot b_{n-k}$ (convolution). For EGFs, the coefficient would be $\sum_{k=0}^{n} \binom{n}{k} \cdot a_k \cdot b_{n-k}$. This intuitively corresponds with, in the OGF case, a generating function with nth coefficients representing the number of pairs of objects with sizes that sum to n from the set of objects represented by f(x) and g(x) (choose k objects from f and n-k from g). The EGF case is similar, but also takes into account the number of ways the objects can be arranged, therefore being more useful for permutation style problems.
- A linear combination of 2 generating functions (for all types of generating functions) is equivalent to the same linear combination of each corresponding coefficients
- For OGFs, multiplying and dividing by (x-c) is utilized to shift coefficient indices, while for EGFs, integration and differentiation are utilized, respectively (this is useful when setting up recurrence relations for generating functions). Therefore, recurrence relations represented by OGFs tend to be functional equations, while recurrence relations represented by EGFs tend to be differential equations.
- Based on these properties, there are many useful compositions that we can implement based on the problem. For example, $\frac{1}{1-f(x)}$ corresponds to choosing a set of objects from the set represented by f(x) in terms of OGFs, and similarly $e^{f(x)}$ for EGFs.

Generally generating functions are set up either by starting from known Taylor Series or starting from a functional/differential equation, after which algebraic (or calculus based in the case of differential equations) manipulations (including the properties mentioned above, especially multiplication of generating functions and compositions) are used to determine the closed, algebraic form of the generating function (sometimes the functional/differential equation itself is sufficient for singularity analysis). From here, we can use differentiation (Taylor Series), Cauchy's Coefficient Formula, partial fraction decomposition (useful for rational generating functions), and other functional decomposition techniques coupled with known Taylor series to find exact forms for desired coefficients.

In many cases, we can find the desired generating function of interest, but we do not have a good way of finding the coefficients (perhaps we can calculate them, but not in O(1) time). In these cases, analytic combinatorics becomes very useful. Unfortunatley, we do not have enough space to go into specific examples/practice problems of generating functions, especially since they are a very famous and well-known concept. Some examples for the reader to try out if interested are general solutions to recurrence problems that are either linear or linear save for a convolution of prior terms (including fibonacci numbers), Catalan numbers (via Segner's reccurrence relation for Dyck words), the fruit basket problem (given some rules about how fruits can be placed into a basket, determine the number of baskets with n fruits in the interior), and zag-numbers (the number of alternating permutations, or permutations such that each number is alternatingly greater than or less than the previous), though there are many more applications, especially regarding counting special types of trees. Again, asymptotics are essential when the generating function coefficients do not have an O(1) closed form solution.

2.4. **Stirling's Formula.** Stirling's formula is one of the most important asymptotics related to combinatorics, which, unlike the general theme of complex analysis, can be derived purely from real analysis techniques.

Stirling's formula gives an O(1) asymptotic for the factorials.

Theorem 2.1 (Stirling's Formula)**.**

$$n! \sim \sqrt{2\pi n} \cdot (\frac{n}{e})^n$$

Proof. We will provide a rigorous proof in one of our examples in section 6 , but for now, an elementary proof of the asymptotic can be found at [MS15]. A summary of the idea is to determine a bound for $|ln(x+1)-x|$ utilizing Taylor series, and with the help of this bound, express $\ln|n!|$ as the sum of $\int_{1/2}^{n+1/2} \ln x \cdot dx$ and $\sum_{k=1}^{n} \ln k - \int_{k-1/2}^{k+1/2} \ln t \cdot dt$. Then, utilize the fact that $\int \ln x dx = x \ln x - x$ and the bound mentioned above to approximate the first integral with error $O(1/n)$ and show that the second integral converges to a constant with error $O(1/n)$ with a similar approach (utilizing the integral of lnx, Taylor series, and the bound, as well as comparison with the integral of $\frac{1}{x^2}$). This gives the asymptotic form in 2.1, save for the coefficient of $\sqrt{2\pi}$, which is determined by considering the integrals $I_n = \int_0^{\frac{\pi}{2}} \sin^n(x)dx$, clearly satisfying $I_n \leq I_{n-1} \leq I_{n-2}$, the recurrence $I_n = \frac{n-1}{n} \cdot I_{n-2}$ (from integration by parts) with initial terms $I_0 = \pi/2, I_1 = 1$, therefore the limit $\lim_{n\to\infty} I_n/I_{n-2} = \lim_{n\to\infty} I_n/I_{n-1} = 1$, and the exact forms $I_{2n} = \binom{2n}{n} \cdot \frac{\pi}{2^{2n+1}}, I_{2n+1} = \frac{2^{2n} \cdot n!^2}{(2n+1)!}$ (verifiable by induction). These forms can be substituted into the limit involving I_n, I_{n-1}, and after substituting Stirling's formula with the unknown coefficient as a variable and solving, we find that it is equal to $\sqrt{2\pi}$. ■

Stirling's formula is especially useful for the sequence encoded by the coefficients of EGFs, as we need to multiply the coefficients by $n!$ to get the EGF sequence values, and all analytic combinatoric methods serve to find asymptotics for the coefficients rather than the EGF sequence values themselves (in a way, analytic combinatorics assumes that all univariate generating functions are ordinary).

2.5. **Introductory Complex Analysis.** The last neccessary subject in order to understand the field of analytic combinatorics is complex analysis. The main idea of analytic combinatorics is to treat generating functions as complex functions that are analytic in a disk of convergence centered at the origin. Then, we can use tools of analysis to analyze functions at their singularities on the boundary of convergence of their Maclaurin expansions (the approach in the case of Meremorphic Asymptotics and Singularity Analysis), or, in the case of Saddle Point Asymptotics, using Cauchy's Integral Formula with the contour over a saddle point, approximating the coefficients with the integral of a normal distribution. As of right now, the reader is not expected to be familiar with all of this terminology, but we will introduce all the necessary concepts in this section. Much of the following theorems can are overviewed in [SS10], so the reader is encouraged to take a look if further reading would be helpful. Additionally, this video: (153) Complex integration, Cauchy and residue theorems — Essence of Complex Analysis #6 provides a great overview of contour integration and Cauchy's Coefficient Formula. A complex functions $f(z)$ is $\mathbb{C} \to \mathbb{C}$, meaning that it is a mapping from one point on the complex plane to another. Therefore, one common way of visualizing complex functions are as a transformation of 2d space, where each point z on the complex plane maps to $f(z)$. This can be thought of as similar to what 2d matrices represent, however, although matrices represent linear transformations of space, complex functions represent all possible transformations of 2d space. A complex function is holomorphic, or differentiable, at a point iff its derivative at the given point exists, defined as

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, h \in \mathbb{C}.$$

Notice that this looks very similar to the definition of a derivative for real functions, the difference being that, for real functions, we can only approach a point from the left or right, while in complex analysis, one can appraoch the point from any direction on the complex plane. If a function is holomorphic, then for small Δz ($\Delta z \to 0$), $f(z + \Delta z) \approx f'(z) \cdot \Delta z + f(z)$. Since $f'(z)$ is a complex number, let $f'(z) = re^{i\theta}$ and $\Delta z = ce^{i\alpha}$ (clearly, $c \to 0$). Then $f(z + \Delta z) \approx rce^{i(\theta+\alpha)} + f(z)$, which clearly shows us that, thinking about f as a transformation of 2d space, $f'(z) = re^{i\theta}$ implies that, close to z, f can be understood as a translation sending the point z to $f(z)$, followed by a linear transformation that can be decomposed into a rotation of angle θ and a scaling of r (in fact, all transformations of space, when approximated close to a point, can be decomposed into a translation followed by a linear transformation, but since in the case of holomorpic functions this transformation is a rotation followed by a scaling, the matrix representing the linear transformation (called the Jacobian matrix in the general case of all transformations of space) is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$). The Jacobian matrix (again, which corresponds to the linear transformation which approximates the transformation represented by a $\mathbb{R}^n \to \mathbb{R}^n$ function around a point after translation) for a 2d vector function $f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} p(x,y) \\ q(x,y) \end{bmatrix}$ is

$$\begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{bmatrix},$$

but if the f represents a complex function (so we can write f as $f(x+yi) = p(x,y) + iq(x,y)$), and this f is holomorphic, then

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \ and \ \frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y}.$$

The above is a pair of extremely important equations known as the Cauchy Reimann equations. One other possible representation of a complex function is as a Polya vector field, where if $f(a+bi) = c+di$, then the vector situated at $a+bi$ is $\begin{bmatrix} c \\ -d \end{bmatrix}$. The Cauchy Riemann equations show that the divergence and the curl of this vector field are both 0. By the Divergence Theorem and Green's Theorem, the flux through any closed curve and and the work done by the vector field on any point that travels one loop around the closed curve are equivalent to the double integral of the divergence and the curl within the curve, respectively. Since both of these are 0, the flux and work done are both 0, an important result when we discuss contour integration.

The last notion we need to discuss before beginning contour integration is analyticity. Similar to real functions, a complex function is analytic if it can be represented by a convergent power series with complex coefficients that converges to the function on a disk with a positive radius of convergence. Namely, $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$, and evidently, $a_k = \frac{f^{(k)}(z_0)}{k!}$ by repeated differentiation of the power series ($a_k \in \mathbb{C}$). The Cauchy Hadamard theorem states that the power series converges in the interior of a disk with radius

$$r = \frac{1}{\limsup_{n\to\infty} a_n^{\frac{1}{n}}},$$

which can be shown by comparing the power series to an infinite geometric series to prove convergence and utilizing the general term test for divergence. We will soon see that a complex function is analytic iff it is holomorphic, which is a surprising result, but the proof requires contour integration, so we will provide the proof in that section. A related result is that a function is infinitely differentiable if it is holomorphic, which can be seen because, considering $f(x+iy) = p(x,y) + iq(x,y)$, $f'(x+iy) = p_x + iq_x$ (seen by approching $x+iy$ from a point to the right/left), which $= q_y - ip_y$ (seen by approaching $x+iy$ from a point above/below). Notice that this gives an alternate proof of the Cauchy Reimann equations by equating, but we can also utilize this to compute the 2nd derivative: $f''(x+iy) = q_{xy} - ip_{xy}$ by differentiating the 1st derivative of f from the first expression above $(p_x + iq_x)$ and approaching from the imaginary direction, and from the second expression for $f' = (q_y - ip_y)$ while approaching from the real direction, we get that $f''(x+iy) = q_{yx} - ip_{yx}$. Since $q_{xy} - ip_{xy} = q_{yx} - ip_{yx}$, we have shown that the derivative of f' is the same no matter whether one approaches $x+iy$ from the right/left or up/down. We can intuitively reason that this should imply that f'' taken from all directions should be equal, but as this involves multivariate limit, it takes more rigor to prove that this is actually the case, which we will not elaborate on here. Therefore, f, f' exists is sufficient to show that f'' exists. Therefore, since f', f'' exist, f''' exists, and therefore it can be shown by induction that all derivatives of f exist.

Now, we can begin one of the most important topics in complex analysis, which is contour integration. The contour integral of a complex function $f(z)$ over a contour $\gamma(t)$ ($\mathbb{R} \to \mathbb{C}$, with $t \in [a,b]$) is written as

$$\int_\gamma f(z)dz,$$

which by substitution, can be written as $\int_a^b f(\gamma(t))\gamma'(t)dt$, which is an integral of a complex function over a real domain (allowing for computation in terms of real integration due to the fact that $\int_a^b \alpha(x) + i\beta(x)dx = \int_a^b \alpha(x)dx + i\int_a^b \beta(x)dx$). Notice that $Re(f(\gamma(t))\gamma'(t))$ is the dot product of the Polya vector situated at $\gamma(t)$ and the vector defined by $\begin{bmatrix} Re(\gamma'(t)) \\ Im(\gamma'(t)) \end{bmatrix}$, while $Im(f(\gamma(t))\gamma'(t))$ is the dot product of the Polya vector field situated at $\gamma(t)$ and the vector defined by $\begin{bmatrix} Im(\gamma'(t)) \\ -Re(\gamma'(t)) \end{bmatrix}$, or the vector perpendicular and oriented $\pi/2$ radians clockwise from $\gamma'(t)$'s vector. This shows that the real part of the contour integration is the work done by the Polya vector field along the contour, while the imaginary part is the flux from left to right. By Green's and the Divergence Theorems, since the divergence and curl of the Polya vector field of a holomorphic function are 0, the contour integral over a closed contour of a function holomorphic inside the contour is 0, which is a very surprising result that highlights the beauty of complex analysis.

Breaking a closed contour in a holomorphic function into two parts and reversing one part, it can be seen that a contour integral between 2 points over a domain in which the function is holomorphic is equivalent to the integral between those 2 points over any contour that only covers point on the function's analytic domain, allowing us to deform the contour in any way we desire. All contributions to the integral come from points where the function is not analytic, called singularities (which we will discuss further), so by deforming the contour, we see that the closed contour integral around a region is the sum of the residues of the singularities within the region, where a residue of a singularity is the value of the contour integral only surrounding that region. Generally, closed contour integrals are taken counter clockwise. Furthermore,

$$\int_\gamma \frac{dz}{z^n},$$

for γ surrounding the origin in a counter clockwise orientation, is equivalent to...

$$\begin{aligned} &\int_0^{2\pi} e^{-ni\theta} \cdot ie^{i\theta} d\theta \\ &= i\int_0^{2\pi} e^{-(n-1)i\theta} d\theta \\ &= \begin{cases} 2\pi i, (n=1) \\ -\frac{1}{n-1}\cdot(1-1) = 0, n \in \mathbb{Z}, n > 1 \\ 0, n \in \mathbb{Z}, n \leq 0 \end{cases} \end{aligned}$$

This evaluation of the contour integral is very useful for what we will discuss next: Cauchy's Coefficient Formula.

Cauchy's Coefficient Formula, for an function $f(z)$ analytic in and on a disk with radius r around the origin. Then the coefficients of the Maclaurin expansion, a_n (corresponding to x^n), satisfy $a_n = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(z)}{z^{n+1}} \cdot dz$, where γ is a closed, counter-clockwise oriented curve inside the disk where f is analytic (for example, the counter-clockwise circle with radius r centered at the origin). This follows by plugging in the Maclaurin expansion of f, and is an extremely important result in analytic combinatorics. This formula can be extend to find the Laurent Series of f if there are pole singularities at the origin by plugging in negative n. A similar result is Cauchy's Integral Formula, which by the same reasoning, asserts that $f(z) = \int_\gamma \frac{f(w)}{w-z} \cdot dw$.

We can now utilize Cauchy's Integral Formula to prove that all holomorphic functions are analytic. Since we have shown that holomorphic functions are infinitely differentiable, it suffices to prove that the power series of a holomorphic function always either converges to the function itself or diverges.

Theorem 2.2. *All complex functions that are holomorphic are also analytic.*

Proof.

$$f(z) = \int_\gamma \frac{f(w)}{w-z} \cdot dw = \int_\gamma \frac{1}{w-a} \cdot \frac{1}{1-\frac{z-a}{w-a}} \cdot f(w)dw$$

$$= \sum_{n=0}^{\infty}(z-a)^n \cdot \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \cdot dw = \sum_{n=0}^{\infty} b_n \cdot (z-a)^n,$$

where b_n are the coefficients of the Taylor Series expansion around a, satisfying $b_n = \frac{f^{(n)}(a)}{n!}$. ■

Now, we will introduce the notion of analytic continuation. Given a function $f(z)$ which is analytic in a given domain, an analytic continuation of f, $g(z)$, is a function such that $f = g$ on the intersection of the domains where they are analytic, and $g(z)$ also extends to a larger domain in which it is analytic. Essentially, g is analytic in a larger domain and equals f when f is analytic. Interestingly, we can prove that two analytic continuations of f are equal on the domain in which they intersect, which essentially means that analytic continuation is a unique, deterministic process. This can be shown by taking Taylor series expansions of f close to the edge of the domain on which it is analytic, therefore extending the domain in a deterministic way, and repeating this process. A singularity is a point that no analytic continuation can include in its domain. For example, $\frac{1}{(z-a)^n}$ for positive integral n is a pole singularity which multiplicity n at a. Other types of singularities include the natural log and fractional powers. These two types of singularities, along with others, requite a branch cut, which is a half line stemming from a singularity which is defined to not be analytic, as two equivalent definitions of the function clash there (for example, $\sqrt{z}$, where argument is considered to be between $-\pi$ and π, is extended to the complex plane by halving the argument of any complex z and taking the square root of the magnitude. At any point on the negative real line, the argument may be defined as $-\pi$ or π, resulting in 2 different evaluations of the square root function (giving a plus or minus), so we add a branch cut on this ray).

We present 2 important theorems for Analytic Combinatorics that allow us to easily find the radius of convergence of a power series, which, when combined with Cauchy-Hadamard's theorem as well as techniques from Meremorphic/Singularity Analysis, gives important information about the coefficients.

Theorem 2.3. *There is at least 1 singularity on the boundary of the disk of convergence of a complex power series.*

Proof. Let the radius of convergence be r, and the power series be $f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$ Evidently, there can be no singularity in the interior of the disk of convergence, as this point is not analytic/holomorphic, but differentiating the power series term-by-term and applying the Cauchy-Hadamard theorem, we see that the radius of convergence is the same, so the power series must be holomorphic inside the boundary of convergence - a clear contradiction. If, instead, the closest singularity were outside the boundary of convergence (with distance d), then analytically continue the power series to have a radius of convergence of r_0 satisfying $r < r_0 < d$. Let r_1 satisfy $r < r_1 < r_0$. Then we are justified in utilizing Cauchy's Coefficient Formula with circular contour of radius r_1. $a_k = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(z)}{z^{k+1}} dz \leq \frac{M}{r_1^k}$, where $M = \sup_{z\in\gamma} |f(z)|$. By Cauchy-Hadamard's Theorem, r (the radius of convergence) $= \frac{1}{\limsup_{k\to\infty} a_k^{(1/k)}} \geq \frac{1}{\limsup_{k\to\infty}(M/(r_1^k))^{(1/k)}} = \frac{r_1}{\limsup_{k\to infty} M^{(1/k)}} = r_1$, or $r_1 \leq r$, which is a contradiction, since we chose $r_1 > r$. Therefore, the nearest singularity to the center of the Taylor Series must be on the circle bounding the disk of convergence of the power series. ■

Theorem 2.4. *Vivanti-Pringsheim Theorem If a complex function's Maclaurin Series coefficients are all non-negative, then there is a dominant (closest or one of the closest) singularity on the positive real line.*

Proof. Let the function be $f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$, and let the radius of convergence be R. For the sake of contradiction, assume that the analytic continuation of $f(z)$ is analytic at $z = R$, and the Taylor Series centered at $z = R$ has a radius of convergence of r. Choose h between 0 and $\frac{r}{3}$, and also smaller than R. Then, by differentiating the Maclaurin Series of f term by term, it can be see that the value of $f(R-h), f'(R-h)$, and all higher order derivatives at $R-h$ are positive. By Theorem 2.3, the radius of convergence of the Taylor Series of f centered at $z = R-h$, is at least $2h$, so this Taylor Series converges at, for example $z = R+h/2$, while the Maclaurin Series diverges when evaluated at this point. However, the Maclaurin Series can be turned into the Taylor Series centered at $z = R-h$ by substituting in $z = (z-(R-h)) + (R-h)$. If we consider $z \in \mathbb{R}$ and $z > R-h$, then both terms in this sum are positive. We can use the binomial theorem within the Maclaurin expansion for f, and since all the terms are positive, we can re-order the terms to form the Taylor Series around $z = R-h$. Therefore, working backwords, we can show that the Maclaurin expansion for f evaluated at $z = R+h/2$ converges, which is a contradiction because R is the radius of convergence. ■

Theorem 2.4 is quite useful in Analytic Combinatorics, as the generating functions generally have Maclaurin Series with non-negative coefficients (as the coefficients generally represent discrete amounts/quantities). We can therefore find the radius of convergence of the Maclaurin Series of a generating function with non-negative coefficients by searching the positive real line and finding the closest singularity, the distance to which providing the radius of convergence.

The last notion we will introduce in this section is that of the Gamma function, generally defined by the integral formula $\Gamma(z) = \int_0^{\infty} t^{(z-1)} e^{-t} dt$. The Gamma function is continuous representation of $(n-1)!$, extended to the complex plane, and though only converges for $Re(z) > 0$, can be analytically continued to the entire complex domain, except for poles at 0 and the negative integers. The Gamma function appears in many asymptotic estimates, especially in Singularity Analysis, and there are many different definitions of the gamma function - the ones most relevant to this paper are the integral form, the Hankel form, the Weistrass form, and the Euler form (we will discuss the latter 3 later). With this background in Complex Analysis, Stirling's Formula, Generating Functions, and Asymptotics, we may begin the most simplest and convenient approach to determining asymptotics: Meremorphic Asymptotics.

3. Meremorphic Asymptotics

3.1. **Exponential Scale.**

First, we present an important theorem for all asymptotic methods:

Theorem 3.1. *[FS09] For a complex function $f(z)$ analytic at the origin with power-series representation $f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$, and dominant (closest to the origin) singularity at radius r (equivalently, a power-series radius of convergence of r)*

$$a_k = (1/r)^k \cdot \alpha(k),$$

where $\alpha(k)$ is a subexponential factor, meaning that $\limsup_{n\to\infty} |\alpha(n)|^{(1/n)} = 1$*, or for any* $a, b > 0$*, we have that* $\alpha(k) = o((1+a)^k)$ *and* $(1-b)^k = o(\alpha(k))$*.*

Proof. Cauchy-Hadamard's theorem states that r (when considered the radius of convergence) satisfies $r = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$. From here, the theorem follows quickly by taking the reciprocal of both sides and substituting in $a_n = (1/r)^n \cdot \alpha(n)$ for any function $\alpha(n)$. Cauchy's-Hadamard's formula is proven by considering the convergence of the power series, because as long as the power series converges, we know that it converges to f since it is holomorphic in this region. If $r > \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = r_0$, then the absolute value of the nth term of the power series sum evaluated at some point z such that $r_0 < |z| = r_1 < r$ is asymptotically equivalent to $(r_1/r_0)^n \cdot \alpha(n) > 1 > 0$, so by the general term test for series convergence, the series diverges (which is a contradiction, as the series must converge within the disk of convergence). If $r < r_0$, then for all points such that $|z| < r_0$, including points outside the radius of convergence, the series converges by the limit comparison test (the absolute values of the terms of the series, if $|z| = r_1$, are $(r_1/r_0)^n \cdot \alpha(n) = o((\frac{r_1}{r_0} + \epsilon)^n)$ for some $\epsilon > 0$, which can be chosen is such a way such that $r_1/r_0 + \epsilon < 1$ because $r_1/r_0 < 1$, meaning that the terms are asymptotically bounded above by a convergent infinite geometric series). This is also a contradiction, because this means that the series is convergent in a disk with radius larger than the radius of convergence, meaning that the disk represented by the radius of convergence is not the largest possible disk inside which the series converges, which violates the definition. Therefore, we have shown that $r = r_0$, completing the proof of Cauchy-Hadamard's theorem, and therefore proving the overall theorem.

■

Theorem 3.1 shows that the location of singularities is essential to determining the exponential scale of asymptotics of coefficients. After the location of the singularities are determined, then the specific sub-exponential factors are determined by analysis of the types of singularities themselves, utilizing the methods presented in this paper. Lastly, before we begin meremorphic asymptotics, we will introduce an interesting result related to exponential scale of coefficients to determine bounds for coefficients.

Theorem 3.2. *[FS09] For a generating function f with non-negative coefficients, the coefficient a_n ($f(z) = \sum_{n=0}^{\infty} a_n z^n$) satisfies $a_n \leq \frac{f(r)}{r^n}$ for any real $r \in (0, R)$, where R is the radius of convergence of the Maclaurin representation of f. Also, since this holds for any r, this upper bound is minimized when r is chosen as $J^{-1}(n)$, where $J(r) = \frac{f'(r)r}{f(r)}$ (and in fact, any approximation for this value of r will result in a valid upper bound).*

Proof. By Cauchy's coefficient formula, $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz$, where in this case, γ is chosen as the counter-clockwise oriented circle centered at the origin with radius r. Therefore,

$$\begin{aligned} & a_n \\ =& |a_n| \\ \le& |\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz| \\ =& \frac{1}{2\pi}\int_\gamma \frac{sup_{z\in\gamma}(|f(z)|)}{r^{n+1}}dz \\ =& \frac{1}{2\pi}\int_\gamma \frac{f(r)}{r^{n+1}}dz \\ =& \frac{2\pi r\cdot f(r)}{2\pi\cdot r^{n+1}} \\ =& \frac{f(r)}{r^{n+1}}. \end{aligned}$$

Since this holds for any $r \in (0, R)$, we can minimize this bound by differentiating with respect to r and setting it to equal 0, resulting in the equation $f'(r)r = nf(r)$, or $n = f'(r)r/f(r)$ (if there are multiple solutions (there must be finitely many), the value of r that minimizes $f(r)/r^{n+1}$ should be chosen, which can easily be determined by testing), resulting in the expression in the theorem statement. ■

3.2. **Meremorphic Asymtotics.** Now that we have established the preliminaries, we may begin our first, most simplest case of determining asymptotics for coefficients of generating functions. Meremorphic functions are functions that are a quotient of two analytic functions (where the denominator function has points that evaluate to 0). Meremorphic functions generally are meremorphic for the entire complex plane, but the techniques developed in this section can also be applied for functions that are meremorphic for $|z| \le r$, assuming that there is a pole singularity within this domain. In essence, meremorphic asymptotics is concerned with the asymptotics for pole-type singularities. Recall that rational functions can be decomposed by partial fraction decomposition, and by taking the Maclaurin series of the summands after decomposition, the coefficients of the rational function itself can be expressed as an exponential polynomial, or rather, a sum of summands of the form of a polynomial multiplied by an exponential function. For example, consider the generating function $f(x) = \frac{x^3-x+3}{x^5-13x^4+67x^3-171x^2+216x-108}$. By partial fraction decomposition, $\frac{x^3-x+3}{x^5-13x^4+67x^3-171x^2+216x-108} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3}$, so $A(x-2)(x-3)^3 + B(x-3)^3 + C(x-2)^2(x-3)^2 + D(x-2)^2(x-3) + E(x-2)^2 = x^3-x+3$, and by differentiation, $A(x-3)^3 + 3A(x-2)(x-3)^2 + 3B(x-3)^2 + 2C(x-2)(x-3)^2 + 2C(x-2)^2(x-3) + 2D(x-2)(x-3) + D(x-2)^2 + 2E(x-2) = 3x^2-1$, and $6A(x-3)^2 + 6A(x-2)(x-3) + 6B(x-3) + 2C(x-3)^2 + 8C(x-2)(x-3) + 2C(x-2)^2 + 2D(2x-5) + 2E = 6x$ Solving (by plugging in $x = 2$ and $x = 3$ into the equations), $A = -38, B = -9, C = 10, D = -28, E = 27$, or $f(x) = \frac{19}{1-(x/2)} - \frac{9/4}{(1-(x/2))^2} - \frac{10/3}{1-(x/3)} - \frac{28/9}{(1-(x/3))^2} - \frac{1}{(1-(x/3))^3}$, so taking the Maclaurin expansion, the Maclaurin series coefficient of x^n of $f(x)$ is $19\cdot(1/2)^n - (9/4)\cdot(n+1)\cdot(1/2)^n - (10/3)\cdot(1/3)^n - (28/9)\cdot(n+1)\cdot(1/3)^n - (n+1)(n+2)\cdot(1/3)^n$ (a similar analysis would still hold even if some of A, B, C, D, E were complex). This is a common approach elaborated on in most single variable calculus classes or classes that cover generating functions, but this approach can be generalized to determine asymptotics for any meremorphic function's coefficients, making it a very powerful tool. The idea is, assuming all the singularities within a radius r of the origin are poles, approximate the function with a rational function with the same approximate pole locations, types, and scaling, and use this rational function's Maclaurin series as the asymptotic for the original meremorphic function's coefficients. We will elaborate on this in much more detail below.

Theorem 3.3. *[FS09] Given a generating function $f(z)$, which has no singularities in the complex plane on the circle centered at the origin with radius r, and only pole singularities within this circle, assume that there are j poles inside, and the kth pole is located at p_k. Assume that the pole is of multiplicity β_k, and the Laurent series expansion of the f at p_k is $f(z) = F(z) + \sum_{m=1}^{\beta_k}(\frac{c_{k,m}}{(z-p_k)^m})$, where F is an analytic function at p_k (notice that all meremorphic functions, or functions that are a quotient of 2 analytic functions, satisfy these premises). Then the coefficient of z^n in the Maclaurin expansion of f, a_n, satisfies*

$$a_n \sim \sum_{k=1}^{j}\sum_{m=1}^{\beta_k}\frac{c_{k,m}}{(-1)^m(p_k)^{m+n}}\cdot\binom{n+m-1}{n} = h(n)$$

(this is an exponential-polynomial on n) with error $\epsilon_n = |h(n) - a_n| \leq \frac{M}{r^n}$ where $M = \sup_{z\in\gamma}|f(z)|$, where γ is the contour of the counterclockwise oriented circle with radius r surrounding the origin (note that $M = f(r)$ if the coefficients of the Maclaurin expansion of f are positive).

Proof. Notice that, if we approximate $f(z)$ with

$$Q(z) = \sum_{k=1}^{j}\sum_{m=1}^{\beta_k}(\frac{c_{k,m}}{(z-p_k)^m}),$$

then considering the Laurent series at original "singularity" location of $f(z) - Q(z)$, we see that the singularity was removed, and the function is now analytic inside and on the circle with radius r centered at the origin. By bounding Cauchy's Coefficient Formula, the error of $\frac{M}{r^n}$ follows. To finish the proof, we simply need to show that the coefficient of z^n in the Maclaurin Series expansion of $Q(z)$ is

$$h(n) = \sum_{k=1}^{j}\sum_{m=1}^{\beta_k}\frac{c_{k,m}}{(-1)^m(p_k)^{m+n}}\cdot\binom{n+m-1}{n},$$

which quickly follows by taking the Maclaurin expansion of each

$$(\frac{c_{k,m}}{(z-p_k)^m})$$

term through repeated differentiation. ■

We will illustrate an example regarding alternating permutations (mentioned in section 2), which is a great example of the power of meremorphic asymptotics. As an important note, we generally do not use the formula provided in Theorem 3.3 directly. Instead, we usually re-derive the formula in the case of the problem, after which we determine the asymptotic for this case. Still, this formula is useful as a way to summarize the key ideas of Meremorphic Asymptotics.

3.3. **An Illustrative Example - Tangent Numbers.** This example is covered in [FS09] For odd n, consider the number of permutations of size n such that each element is alternatingly greater than or less than the previous (let this be a_n) (we start with an increase (up-down)). For example, if n = 5, then 2, 4, 1, 5, 3 is an alternating permutation. One can see that the number of down-up alternating permutations is equivalent to the number of up-down alternating permutations by subtracting each element from $n+1$, so we can get the recurrence $2a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k}$ by iterating on the location of the largest element. If $f(x)$ is the EGF, $2\frac{df}{dx} = (f(x))^2 + 1$ (the +1 because $2 * a_1 = 2 = 1 + a_0^2 = 2$) Solving this differential equation by separation of variables with I.C. $a_0 = 1$, $\frac{t}{2} + C = \int \frac{df}{f^2+1}$, so we get

$$f(x) = tan(x/2 + \pi/4) = tanx + secx.$$

Since n is odd, we care about the coefficients of the Maclaurin expansion of

$$tan(x),$$

after which, we multiply the nth coefficient by $n!$.

Notice that $tan(a+bi)$ only has singularities when $cos(a+bi) = cos(a)cos(bi) - sin(a)sin(bi) = (e^b + e^{-b})cos(a)/2 - i(e^b - e^{-b})sin(a)/2 = 0$, which only occurs at $a + bi = \pi/2 + k\pi$ for any $k \in \mathbb{Z}$. These singularities are poles, since sin and cos are entire (analytic on all $\mathbb{C}$), so meremorphic asymptotics applies. $sin(\pi/2) = 1$, and $cos(x) = cos((x - \pi/2) + \pi/2) = -sin(x - \pi/2) = -(x - \pi/2)(1 + (x - \pi/2)^2 F(x - \pi/2))$, where F is an analytic function at 0. Therefore, $tanx = \frac{-1}{x-\pi/2} + G_1(x - \pi/2)$ for some analytic G_1 at 0, and similarly $tanx = \frac{-1}{x+\pi/2} + G_2(x + \pi/2)$ for analytic G_2. The rational approximation for $tanx$ at these singularities is $\frac{2/\pi}{1-x\cdot 2/\pi} - \frac{2/\pi}{1+x\cdot 2/\pi}$. Therefore, $\frac{a_n}{n!} \sim 2 \cdot (\frac{2}{\pi})^{n+1}$ for odd n, or by Stirling's Formula, $\boxed{a_n \sim \frac{4\sqrt{2\pi n}}{\pi} \cdot (\frac{2n}{\pi e})^n}$, with a possible error bound $\frac{M}{\pi^n}$, where $M = \sup_\gamma tan(z)$ for γ being the circle centered at the origin with radius π. The value of M can be found by replacing z by $\pi e^{i\theta}$, expanding tan with the angle sum formula, differentiating with respect to θ, and setting this derivative to be equal to 0. We will not go into the specifics here, as the computations are a bit tedious, but the general idea is hopefully clear to the reader at this point.

4. Singularity Analysis and Related Methods

The main idea of singularity analysis is as follows: so far, we have a very powerful tool to generate asymptotics of functions with pole singularities, namely Meremorphic Asymptotics. Unfortunately, many generating functions that are relevant in combinatorics do not have pole singularities, so it is necessary to develop methods to analyze singularities of different kinds, as well as to improve on the error estimates for these analyses. In Singularity Analysis, we utilize techniques from complex analysis to derive formulas for asymptotics of other, specific classes of singularities, for example, z^α for non-integral α or lnz. We then apply these results to generating functions who behave approximately close to a studied type of singularity at their singularities, and we are therefore able to determine asymptotics for these functions' coefficients.

4.1. **Singularity Analysis Main Theorems.** Perhaps the most central theorem in Singularity Analysis is the analysis of the function $f(z) = (1 - z)^{-\alpha}$, for $\alpha \in \mathbb{C}$. The theorem is as follows (note that we denote the coefficient of z^n in the Maclaurin Series expansion of $f(z)$ as $[z^n]f(z)$)...

Theorem 4.1. *[FS09]* $[z^n](1-z)^{-\alpha} = \frac{\prod_{k=0}^{n-1}(\alpha+k)}{n!} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \cdot (1 + \sum_{k=1}^{j-1} \frac{e_k}{n^K}), \alpha \in \mathbb{C}, e_k = \sum_{l=k}^{2k} \gamma_{k,l} \prod_{m=1}^{l}(\alpha - m), \gamma_{k,l} := [v^k t^l] e^{-t}(1-vt)^{-1-\frac{1}{v}}$ *with error* $O(\frac{1}{n^j})$.

Proof. The proof of this theorem is very beautiful, involving contour integration and equivalence of definitions of the gamma function.

Before we begin the proof of the asymptotic, notice that the equivalence $[z^n](1-z)^{-\alpha} = \frac{\prod_{k=0}^{n-1}(\alpha+k)}{n!}$ follows directly from taking the Maclaurin Series. This equivalence is included in the theorem statement solely for the purpose of illustrating that, although we do have a convenient formula for the Maclaurin series of $(1-z)^{-\alpha}$, this formula is computable in $O(n)$ time rather than $O(1)$, making asymptotics useful.

Part 1: Arriving at Hankel's Definition of the Gamma Function [FS09] We begin with Cauchy's Coefficient formula, stating that $[z^n](1-z)^{-\alpha} = a_n = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(z)dz}{z^{n+1}}$. Choose γ to be the contour that is a circle with radius R (we will make $R \to \infty$ to simplify our contour), save for a notch that returns and loops around the (usually) singularity at $z = 1$, essentially being the clockwise oriented curve from the perspective of $z = 1$ that is always at a distance of $\frac{1}{n}$ from the real half-line starting at $z = 1$. By substituting in $t = n(z-1)$, we get the integral $\frac{n^{\alpha-1}}{2\pi i} \int_H (-t)^{-\alpha}(1 + \frac{t}{n})^{-n-1} dt$, where H is the Hankel contour that is at a distance from the positive real line of 1 and wraps around the origin clockwise. Notice that $(1 + t/n)^{-n-1}$ is close to the definition of e, so we can extract e^{-t} from this, giving us the expansion $(1+t/n)^{-n-1} = e^{-t} \cdot (1 + \sum_{k=1}^{\infty} \sum_{j=k}^{2k} ([v^k t^j] e^t (1+vt)^{-\frac{1}{v}-1}) \cdot \frac{t^j}{n^k})$, where the coefficients $[v^k t^j] e^t (1+vt)^{-\frac{1}{v}-1}) = [v^k t^j] e^{t - ln(1+vt)(1+\frac{1}{v})}$ can be extracted by differentiation and L'hopital's rule. For now, we will just asymptotically approximate $(1+t/n)^{-n-1}\, e^{-t}$, but we will return to this full expansion form in order to determine asymptotics with better error. This gives us $a_n = \frac{n^{\alpha-1}}{2\pi i} \int_H (-t)^{-\alpha} e^{-t} dt (1 + O(1/n))$. This form allows us to utilize the Hankel definition of the Gamma function to arrive at $a_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)}$, since $\frac{1}{2\pi i} \int_H (-t)^{-\alpha} e^{-t} dt = \frac{1}{\Gamma(\alpha)}$. We will prove this now.

Part 2: Proof of Hankel's Definition of the Gamma Function

Our goal is to prove the formula $\frac{1}{2\pi i} \int_H (-t)^{-\alpha} e^{-t} dt = \frac{\Gamma(1-z) sin(\pi z)}{\pi}$. We will then prove that this is equal to the reciprocal of the Gamma function in part 3. This equality is the same as $\Gamma(z) = \frac{1}{2 i sin(\pi(z-1))} \int_H (-w)^{z-1} e^{-w} dw$ (here, we have reversed the direction of the Hankel Contour to now be counter-clockwise in order to account for the negative in $sin(-\pi z)$). Note that we will consider the reversed version of the Hankel Contour as H from this point to the rest of the part 2. We will also replace the contour H with h, which is essentially the Hankel contour, the only difference being than, rather than being at a distance of 1 from the real half line, the distance is set to approach 0. We can see that the integral over this contour should evaluate to the same value, as adding 2 segments above and below the real half line (oriented perpendicular to the real line) to connect the two contours at an infinite distance from the origin, we see that the combined contour is a closed contour, and in the interior of this closed contour, the integrand is holomorphic, so the entire integral evaluates to 0. Since the segments are taken to approach $+\infty$, based on the definition of the integrand (specifically, the fact that $w^{z-1} \cdot e^{-w}$ approaches 0 as $w \to \infty$), they evaluate to 0, so separating the combined integral into H and h by reversing one of them, we can show that the contour integrals are equal. Consider $\int_h w^{z-1} e^{-w} dw$. We then split up h into the two linear parts and the circular part, the latter of which evaluates to 0 by substitution and bounding the integral by taking the absolute value of the integrand (which approaches 0 as the distance of the contour to the real half line approaches 0). The integral evaluated on the contour line above the real positive half line approaches $-\Gamma(z)$, while the integral below the contour line, by substitution, can be shown to approach $e^{2\pi i(r-1)}\Gamma(z)$. Combining gives us that the integral $= \Gamma(z)(e^{2\pi i(r-1)} - 1)$. If we multiply both sides by $e^{\pi i(r-1)} = (-1)^{r-1}$, after simplification and moving the $(-1)^{r-1}$, we get the desired result: $\Gamma(z) = \frac{1}{2isin(\pi(z-1))}\int_H(-w)^{z-1}e^{-w}dw$. This video: Gamma Function: Hankel Contour Definition gives a great overview of this step of the proof, if further revision is necessary.

Part 3: Proof of Euler's Reflection Formula

[Knu97,Spi65,LS99] Now, we are left with proving that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}$. By looking at the zeros of the sin function, we can get the Weistrass Product $sinx = \pi x \prod_{n\neq 0}(1 - \frac{x}{n})$, since $sinx$ is an infinite polynomial by Taylor expansion. The Weistrass form of the Gamma Function is $\frac{1}{\Gamma(z)} = ze^{\gamma z} \cdot \prod_{n=1}^{\infty}(1 + z/n)e^{-z/n}$, where γ is the Euler-Mascheroni constant. Also, recall that $z\Gamma(z) = \Gamma(z+1)$ (which can be proven by integration by parts). We can algebraically show that $-z\Gamma(-z)\Gamma(z) = \pi/sin(\pi z)$ by substituting in the Weistrass form of the Gamma function and the Weistrass Product for $sinx$, so the result follows by utilization of the above identity for the Gamma function.

Unforutunately, it is outside the scope of this paper to fully prove that the Weistrass definition of the Gamma function is equivalent to the integral definition. The proof we follow is essentially proving that the Weistrass and the Euler definitions of the Gamma function and equivalent (see this link [Knu97]) and then proving that the Euler definition is equivalent to the Integral definition (see ONE NEAT PROOF! Deriving the EULER DEFINITION of the Gamma Function!). The sources provided give satisfying proofs of these results.

Part 4: Full Expansion As an outline for this step, we can utilize the full expansion $(1 + t/n)^{-n-1} = e^{-t} \cdot (1 + \sum_{k=1}^{\infty}\sum_{j=k}^{2k}([v^k t^j]e^t(1+vt)^{-\frac{1}{v}-1}) \cdot \frac{t^j}{n^k})$, and for each term in the sum, we can take the factor of n outside the integral, and we can multiply the t factor (after adjusting the coefficients so it is a $-t$ factor) into $(-t)^{-\alpha}$ factor of the integrand. Then, we get a factor of the form $(-t)^{-\alpha+k}$ for some k, which results in the coefficient being divided by $\Gamma(\alpha - k)$, which can then be turned into $\Gamma(\alpha)$ by the repeated usage of the recursive product identity utilized in part 3 ($z\Gamma(z) = \Gamma(z+1)$). Working out the specifics, we get the full expansion presented in the Theorem statement. ■

We can extend this result to consider even more cases by adding a $(lnz)^{\beta}$ singularity type.

Theorem 4.2. *[FS09]* $[z^n](1-z)^{-\alpha}(\frac{1}{z}ln(\frac{1}{1-z}))^{\beta} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(ln(n))^{\beta} \cdot (1 + \sum_{k=1}^{j-1}\frac{C_k}{(ln(n))^k}) = h(n), C_k = \frac{\prod_{m=0}^{k-1}(\beta-m)}{k!}\Gamma(\alpha) \cdot (\frac{d^k}{ds^k}(\frac{1}{\gamma(s)}))|_{s=\alpha}$ *with error* $O(1/(ln(n))^j)$.

Proof. Let $f(z) = (1-z)^{-\alpha}(\frac{1}{z}ln(\frac{1}{1-z}))^{\beta}$. Then, plugging in to Cauchy's Coefficient Formula with the contour and substitution from the proof of the previous theorem, we get that the integrand is $f(1+t/n)(1+t/n)^{-n-1} \sim e^{-t}(\frac{-n}{t})^{\alpha}(ln(\frac{-n}{t}))^{\beta}$ by substitution for the formula of f and the definition of e. This, by algebraic manipulations and the binomial theorem (if β is a non-negative integer)/Taylor series expansion of $(1 - ln(-t)/ln(n))^{\beta}$ (in all other cases), $= e^{-t}(-t)^{-\alpha}n^{\alpha}(ln(n))^{\beta}(1 - \beta\frac{ln(-t)}{ln(n)} + \frac{\beta(\beta-1)}{2!}(\frac{ln(-t)}{ln(n)})^2 + ...)$, which directly gives the initial asymptotic formula when combined with the previous theorem, or more specifically, the result is $[z^n](1-z)^{-\alpha}(\frac{1}{z}ln(\frac{1}{1-z}))^{\beta} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(ln(n))^{\beta}$. For a full expansion, notice that $\frac{1}{2\pi i}\int_H(-t)^{-s}e^{-t}(ln(-t))^k dt = (-1)^k\frac{1}{2\pi i}\frac{d^k}{ds^k}[\frac{1}{2\pi i}\int_H(-t)^{-s}e^{-t}dt] = (-1)^k\frac{d^k}{ds^k}\frac{1}{\gamma(s)}$. The full expansion then follows from this, combined with Taylor expansion of $(1 - ln(-t)/ln(n))^{\beta}$.

Notice that the full expansion is in descending powers of $ln(n)$, and there are no terms of descending powers of n because $(ln(n))^k$ is always $o(1/n)$. If β is a non-negative integer, then the asymptotic expansion composed of descending powers of $ln(n)$ is finite, so we can utilize a similar approach to the previous theorem to derive a further asymptotic expansion including descending powers of n, allowing for more accurate asymptotics. ■

The special cases described below follow by manipulation of the formula/derivation process: If α is a negative integer k or 0, consider the limit as $\alpha \to k$ (or 0), to see only a slight alteration in the formula. Also, one can derive better error estimate of the scale $O(1/n^m)$ if β is a positive integer, which is relatively simple by following a similar process to the proof of the original asymptotic, but we will not go into details here.

In fact, this result can be generalized even further by the HLK Tauberian Theorem to determine asymptotics, however, utilizing this theorem comes at a cost - namely that one cannot analyze how the error of the asymptotic grows, nor determine more accurate asymptotics (full expansions) that decrease the error growth. Nevertheless, the theorem is still very useful, so we present it here...

Theorem 4.3. *HLK Tauberian Theorem If $A(n)$ satisfies* $\lim_{n\to\infty} A(cn)/A(n) = 1$, *then A is slowly varying, and* $[z^n](1-z)^{-\alpha}A(\frac{1}{1-x}) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}A(n)$.

Proof. This can be proved very similarly to the previous. Specifically, let $f(z) = (1-z)^{-\alpha}(\frac{1}{z}ln(\frac{1}{1-z}))^{\beta}$. Then, plugging in to Cauchy's Coefficient Formula with the contour and substitution from the proof of the previous theorem, we get that the integrand is $f(1+t/n)(1+t/n)^{-n-1} \sim e^{-t}(\frac{-n}{t})^{\alpha}A(-n/t)$, which, by the definition of slowly varying functions, is $\sim e^{-t}(\frac{-n}{t})^{\alpha}A(n)$. Technically, there is more rigor necessary, as generally slowly varying functions are defined with positive c, but this proof is sufficient for our purposes in terms of an intuitive understanding for where the theorem comes from. ■

These theorems, as we will see, form the basis of Singularity Analysis, and when combined with Transfer Theorems, allow for analysis of a wide variety of functions.

4.2. **Singularity Analysis Sim-Transfer and Related Theorems.** There are 2 major theorems in this section that serve as extensions of the previous fundamental theorems. The first, regarding transfers of big O and small o notations, is as follows...

Theorem 4.4. *[FS09] If* $f(z) = O((1-z)^{-\alpha}(ln(\frac{1}{1-z}))^{\beta})$, *then* $[z^n]f(z) = O(n^{\alpha-1}(ln(n))^{\beta})$, *and similarly, if* $f(z) = o((1-z)^{-\alpha}(ln(\frac{1}{1-z}))^{\beta})$, *then* $[z^n]f(z) = o(n^{\alpha-1}(ln(n))^{\beta})$. *A similar result holds for the HLK Tauberian Theorem.*

Proof. This theorem can be proven by considering Cauchy's Coefficient Formula on the contour that has radius $r > 1$ (centered at the origin) for the majority of the loop, but becomes linear when the argument of the points of the contour are θ or $-\theta$. The contour decreases in distance to $z = 1$ in this linear section, before looping around $z = 1$ in a clockwise orientation with distance $1/n$. It is mathematically defined as

$$\gamma = \begin{cases} \gamma_1 = z||z-1| = 1/n, |arg(z-1)| \geq \theta \\ \gamma_2 = z|1/n \leq |z-1|, |z| \leq r, arg(z-1) = \theta \\ \gamma_3 = z||z| = r, |arg(z-1)| \geq \theta \\ \gamma_4 = z|1/n \leq |z-1|, |z| \leq r, arg(z-1) = -\theta \end{cases},$$

where argument is taken to be $\in (-\pi, \pi]$. Then, we can show that the inner γ_1 circle is asymptotically $O(n^{\alpha-1})$ by bounding the absolute value of the integrand in Cauchy's Coefficient formula (and $o(n^{\alpha-1})$) in the other case. The integral over y_3 is evidently $O(r^{-n})$ due to its radius, so we are left with analyzing γ_2 (and an analysis of γ_4 would follow identically). We can substitute in the change of variables for the contour γ_2 to get real integration bounds into Cauchy's Coefficient Formula, and then extend the right hand bound to $+\infty$ to get an upper bound for the integral. We can also further this upper bound by substituting $f(z)$ with $K(1-z)^{-\alpha}ln(\frac{1}{1-z})^{\beta}$, where K is chosen such taht the absolute value of the right hand side is greater than the absolute value of f (which is possible due to the initial condition). We can then bound the absolute value of the integrand, and follow a similar approach to deal with the ln as we did in the proof of Theorem 4.2. We arrive at the asymptotic bound $O(n^{\alpha-1})$ for the first case and $o(n^{\alpha-1})$ for the second. ■

The following theorem follows almost immediately from the previous, and is useful to quickly come up with asymptotic formulas without considering error.

Theorem 4.5. *[FS09] If $f(z) \sim (1-z)^{-\alpha} \cdot (\frac{1}{z}ln(\frac{1}{1-z}))^{\beta}$, then $[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(ln(n))^{\beta}$ and a similar result holds for the HLK Tauberian Theorem*

Proof. This collorary follows directly from the previous bullet and the fact that $f(x)\ g(x) \leftrightarrow f(x) = g(x) + o(g(x))$ ■

Note that, though we will not provide a formal proof, **we can arrive at a similar result [FS09] for** the previous 2 theorems if we have **a function with multiple singularities** of the analyzed type on the circle bounding the disk of convergence, which can be proven by a similar approach to Theorem 4.4, but with a similar contour with multiple incisions rather than one (see the proof for 4.4).

These theorems are enough to approach many example problems, but before we start a practice problem, we will introduce 2 more methods

4.3. **Related Methods.** The first theorem is a result of simple real analysis methods, and can tackle some scenarios that singularity analysis cannot, however, one cannot determine full asymptotic expansions (improving the error asymptotics) for this theorem.

Theorem 4.6. *[FS09] Let a(z) and b(z) be generating functions with radii of convergence α and β, respectively, such that $\alpha > \beta$. Then $[z^n]a(z)b(z) \sim a(\beta)b_n$, where $[z^n]a(z) = a_n$ and $[z^n]b(z) = b_n$.*

Proof. Let $[z^n]a(z)b(z) = \prod_{k=0}^{n} a_k b_{n-k} = b_n(a_0 + a_1(\frac{b_{n-1}}{b_n}) + (\frac{b_{n-2}}{b_{n-1}} \cdot \frac{b_{n-1}}{b_n}) + ...) \sim b_n(\sum_{k=0}^{\infty} a_k \cdot \beta^k) = a(\beta)b_n$, the similarity step being justified because we can show that the error between the full power series and the Taylor approximation approaches 0 as the number of terms approaches ∞ by Lagrange's error bound, as the factorial grows faster than an exponential. Then, we can show that the end tail of the LHS of the asymptotic equivalence also vanishes by similar reasoning (with bounding), so we are simply comparing the first few terms of the LHS and claiming that they are asymptotically similar to the first few terms on the RHS. This follows as long as the number of included terms approaches infinity, but the number of excluded terms also approaches infinity, as in this way, all the included factors of the form b_{n-k}/b_{n-k-1} approach β. ■

Theorem 4.7. *Darboux Method [FS09] If $f(e^{i\theta})$ is continuously differentiable k times (where $f(e^{i\theta})$ is k-times continuously differentiable if its real and imaginary parts are both k-times continuously differentiable in the real sense), then $[z^n]f(z) = o(\frac{1}{n^k})$.*

Proof. This proof is very quick. Start with Cauchy's Coefficient Formula, substitute in the parameterization $\gamma = e^{i\theta}$, and utilize sucessive integration by parts to find that $\frac{1}{2\pi(in)^k}\int_0^{2\pi} f^{(k)}(e^{i\theta})e^{ni\theta}d\theta$, which, by bounding by intergrating over the absolute value of the integrand, is $o(\frac{1}{n^k})$. ■

Now, we will begin an example of Singularity Analysis.

4.4. **An Illustrative Example - An Unusual Extension of Dyck Words. Problem:** Define a Dyck-extension object to be an object of size n which is composed of a Dyck word (a permutation of an equal number of As and Bs where at no point there are more Bs than As) of length $2n-2$ and a permutation of the integers from 1 through n. Find an asymptotic approximation for the number of sets of Dyck-extension objects that have a sum of lengths of n.

Solution: Notice that regular Dyck words are all of the form A"dyck word"B"dyck word," giving Segner's reccurence $c_{n+1} = \sum_{k=0}^{n} c_{k-n}c_k, c_0 = 1$ for c_n as the number of Dyck words of length 2n. The OGF of c_n, satsfies $c(x) = 1 + x(c(x))^2$, and therefore $c(z) = \frac{1-\sqrt{1-4x}}{2x}$, but since we care about the number of Dyck words of length $2n-2$, our generating function is $\frac{1-\sqrt{1-4x}}{2}$. Since we are adding a permutation of n integers to each object, we are multiplying each coefficient by $n!$, so it suffices to treat our function as an EGF. We have to compose our EGF with e^x, as we are considering a set of Dyck-extension objects, so we end up with the EGF $e^{\frac{1-\sqrt{1-4z}}{2}} = f(z)$.

Now, we begin the analytic combinatorics stage. $f(z) = \sqrt{e}\cdot(1 - \frac{\sqrt{1-4z}}{2} + \frac{1-4x}{8} - ...) = -\sqrt{e}\cdot\frac{\sqrt{1-4z}}{2} + F_1(z) + (1-4z)^{3/2}\cdot F_2(z)$, where F_1, F_2 are analytic at the singularity $z = 1/4$. Therefore, we have that the desired coefficient is $\sim 4^n \cdot \frac{-\sqrt{e}}{2}\cdot\frac{1}{n^{1.5}\Gamma(-0.5)} = 4^{n-1}\sqrt{\frac{e}{\pi}}\cdot n^{-1.5}$ with error $O(n^{-2.5})$. Using Stirling's formula, we get that the desired quantity is $\sim 4^{n-1}\cdot n^{-1.5}\cdot\sqrt{2en}\cdot(n/e)^n = \frac{4^{n-1}\cdot\sqrt{2e}\cdot(\frac{n}{e})^n}{n}$.

5. Functional Equations

Often times, in Analytic Combinatorics, we want to determine asymptotics for a generating function representing a useful quantity, however, this generating function can only be defined by a functional equation, and we have no good way of solving this functional equation for an exact form. Fortunately, in many cases, we can derive full asymptotic formulas, or at least the exponential scale, directly from functional equations. Here, we outline 2 examples that allow for determination of full asymptotic formulas to highlight some of the main ideas/approaches.

5.1. **Polya Alcohols. Problem:** [FS09] The number of chemical isomers of alcohols with n carbon atoms without asymmetric carbon atoms has the Ordinary Generating Function $M(z)$, which satisfies

$$M(z) = \frac{1}{1 - zM(z^2)}.$$

We will not go into the specifics on how this is derived, as the main focus is on the analytic aspect of the problem, however, hopefully the reader now sees, at this point in the paper, the wide applicability of generating functions and Analytic Combinatorics. **Solution:** We conjecture that the coefficients satisfy $M_n = K\cdot\beta^n(1 + O(B^{-n}))forB > 1, \beta \approx 1.681, K \approx 0.361$. We remark that the first few terms of $M(z)$ are $1 + z + ...$ We also note that the coefficients of $M(z)$ must be positive due to the combinatorial class they represent. Therefore, one can see that $M(z) \geq \frac{1}{1-z-z^3}$, which follows by replacing $M(z^2)$ with $1 + z^2$ and noticing, by considering the Taylor Series expansion, that adding more terms to the approximate of $M(z^2)$ only increases the size of the coefficients due to positivity. This implies, since this function has a dominant pole at $z = 0.682$, that the dominant pole of M has a lesser distance to the origin. We can construct a similar argument by replacing $M(z^2)$ with $M(z)$ due to the non-negative nature of the coefficients implying that M is an increasing function, so since $z^2 < z$ for $z \in [0, 1]$, then $M(z^2) < M(z)$, so $M(z) \leq \frac{1}{1-zM(z)}$, therefore showing that the singularity exists at a distance to the origin greater than 0.25 due to bounding with the Catalan generating function. We can then follow a similar process to the first step to narrow down a smaller interval for the location of this singularity. Due to the increasing nature of M, $zM(z^2)$ must $\to 1$ as $z \to$ the singularity, ρ, from the left (otherwise, there would be another singularity (pole) inside the disc of convergence, which is a contradiction). One can therefore also use the equation $\rho M(\rho^2) = 1$ to approximate ρ. This limit also implies that there is a pole singularity at ρ, and this pole has degree 1, since $(zM(z^2))'|_\rho = M(\rho^2) + 2\rho^2 M'(\rho^2)$ is positive due to the non-negative nature of the coefficients. Therefore, we can compute the scaling factor, and by Taylor Series manipulation, this turns out to be $\frac{1}{\rho M(\rho^2)+2\rho^3 M(\rho^2)}$. Also, β must $= \frac{1}{\rho}$.

5.2. **Inversions of a Specific Type. Problem:** [FS09] Determine information about the asymptotics of the coefficients of $y(z)$ that satisfies the functional equation $y(z) = z\phi(y(z))$, for ϕ being expandable as a Maclaurin Series with non-negative coefficients. This is essentially asking about the inversion of the function $\frac{u}{\phi(u)}$, where $\phi(u)$ must have non-negative Maclaurin Series Coefficients. **Solution: Part 1 - Exponential Scale:** We claim that the dominant singularity is located at $\rho = \tau/\phi(\tau) = 1/(\phi'(\tau)$, where τ uniquely satisfies the second equality. This gives the exponential scale of coefficients of $(\frac{1}{\rho})^n$. Firstly, note that $y(z)$ has non-negative Maclaurin coefficients, which can be verified by the method of indeterminant coefficients. Therefore, by Vivanti-Pringsheim's theorem, the dominant singularity is located on the positive real half-line. Let r be the radius of convergence, and define $y(r) := \lim_{x\to r^-} y(x)$. We would like to prove that $y(r) = \tau$, the unique solution of the equation $\tau\phi'(\tau)/\phi(\tau) = 1$. Note that $y(z) = \psi^{-1}(z)$ if $\psi(z) = \frac{z}{\phi(z)}$. Since $\phi(0) \neq 0$, and the function has non-negative Maclaurin coefficients (so it is increasing), $\phi(z)$ does not equal 0 for $z \in [0, r]$. Therefore, there can only be a singularity of $y(z)$ at $z = \rho$ if $\psi'(z) = \frac{\phi(z)-z\phi'(z)}{(\phi(z))^2} = 0$, which occurs uniquely at $z = \tau$. This shows that $y(r) = y(\rho) = r$, so by inverting both sides (applying ψ, we get the desired result). **Part 2 - Asymptotic Equivalence (Outline of Solution Method)** We claim that the singular expansion of y(z) near ρ is $y(z) = \tau - d_1\sqrt{1 - z/\rho} + \sum_{j=2}^{\infty}(-1)^j d_j \sqrt{(1 - z/\rho)^j}$, with $d_1 = \sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}}$ and computable d_j. If we are able to prove this, then the asymptotic estimate $[z^n]y(z) \sim \sqrt{\frac{\phi(\tau)}{2\phi''(\tau)}} \frac{\rho^{-n}}{\sqrt{\pi n^3}}(1 + \sum_{k=1}^{\infty} e_k/n^k)$ for computable e_k follows directly from Singularity Analysis. We can prove this easily by repeatedly differentiating $H(y) = \frac{\tau}{\phi(\tau)} - \frac{y}{\phi(y)}$ at τ to get the Taylor Series expansion (it can be shown that $H(y) = H'(y) = 0$ and $H''(y) \neq 0$). Then, we can invert this taylor series expansion to get the square root term, plus a full asymptotic expansion if more care is included in the derivation process. Note that we invert $H(y)$ to get $y(z) - \tau$, as $H(t)$ can be written as $\rho - z$, for z close to ρ, and we want to find y(z) near this points of the form $\rho - z$ in order to analyze the singularity type, as we know that y has a singularity at $z = \rho$.

6. Other Approaches: Saddle Point Asymptotics

6.1. **Saddle Point Asymptotics Outline.** [FS09] The saddle point method follows a completely different approach from the theme of Meremorphic Asymptotics and Singularity Analysis. The general idea is to start from Cauchy's Coefficient Formula, and turn the contour integral into an integral over a real interval of the form $e^{F(z)}$ by substitution. Then, by splitting the contour into 2 parts and choosing the size of these parts appropriately, we can show that one part is asymptotically negligible, and for the other part, which should be centered at a maximum of $F(z)$, we can approximate the function with an incomplete integral of a Gaussian curve (we essentially use the first 2 terms of the Taylor expansion of $F(z)$ at the maximum, assuming the second derivative is non-zero), and we can then show that the tails of the Gaussian distribution have an asymptotically negligible integral, allowing us to add it in and use fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ with scaling to compute an asymptotic.

The 3 sufficient conditions [FS09] for the Saddle Point Method to work are as follows (when we refer to the integral, we are considering the integral that results from substitution from Cauchy's Coefficient Formula to get a real interval of integration).

(1) As n grows, the integral can be split into 2 parts such that 1 part asymptotically grows slower than the other (small o notation), and the part of the integral that is not negligible should include a relative maximum/stationary point of the integrand when traveling along the contour (in terms of magnitude).
(2) The $F(z)$ component in the non-negligible integrand can be expressed as (by Taylor Series expansion around the maximum) a quadratic term plus an infinite polynomial term, the second (infinite polynomial) term approaching 0 as $n \to \infty$, allowing us to approximate the non-negligible component with an incomplete Gaussian integral.
(3) The tails of the Gaussian integral can be completed back (due to being asymptotically negligible).

Conditions 1 and 3 requite the length of the interval of integration in the non-negligible integral to be sufficiently large, which is in conflict with the fact that the length of the interval should be sufficiently small to satisfy condition 2. Therefore, a general heuristic [FS09] can be created to determine the length, θ_0 of the contour: specifically, $\lim_{n\to\infty} f''(\delta)(\theta_0)^2 = +\infty, \lim_{n\to\infty} f'''(\delta)(\theta_0)^3 = 0$.

After determining the value of the radius r and the length of the interval of integration θ_0 based on these heuristics/the outline of the general approach, we must first verify that the 3 conditions are satisfied, rigorously, before we can conclude an asymptotic formula.

The saddle point method is best illustrated with example problems. We will provide one example of Saddle Point Asymptotics applied to finding an asymptotic formula for the inverse factorial, which are the coefficients of the generating function e^z. This is a simple example that is already solved by Stirling's formula, but it provides an outline for the general process that can be applied to other problems in Analytic Combinatorics that cannot be analyzed in a different way. For instance, Saddle Point Asymptotics is a very powerful tool to determine the asymptotics of the number of integer partitions, and a famous unsolved problem regarding integer partitions is how to exactly compute them in $O(1)$ time. As usual, Analytic Combinatorics, specifically Saddle Point Asymptotics in this case, essentially sidesteps the problem and tackles the question of approximating this value. Now, we will begin the following simpler example to showcase the method of Saddle Point Asymptotics.

6.2. **An Illustrative Example - The Inverse Factorial.** [FS09] $[z^n]e^z = [z^n]f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}dz$ (γ is a counter-clockwise oriented circle with radius r,) which, by substitution, $= \frac{e^n}{2\pi n^n} \cdot \int_{-\pi}^{\pi} e^{n(e^{i\theta}-1-i\theta)}d\theta$ (we set $r = n$, because, in this integral, it places the saddle point (where the first derivative of the integrand is 0) on the real line (at $\theta = 0$)). We shift the interval of integration and split it into the dominant part, on the interval $[-\theta_0, \theta_0]$, and the part that tends to 0 $\theta_0, 2\pi - \theta_0$. In order to satisfy the heruistic mentioned in the previous slide, we set $\theta_0 = n^{-2/5}$ (n^α would work for $\alpha \in [-\frac{1}{2}, -\frac{1}{3}]$). By Taylor approximation of $n(e^{i\theta}-1-i\theta)$, and u-substitution, we can transform the integral into an approximate Gaussian integral, which leads us to the asymptotic $[z^n]e^z = \frac{1}{n!} \sim \frac{e^n}{n^n\sqrt{2\pi n}}$. We can then rigorously show that the other integral approaches 0 as $n \to \infty$, that the tails of the Gaussian Integral tend to 0 (so we can add them), and that the non-quadratic part of the Taylor Expansion approaches 0 when considered in the integral. In fact, by substitution, taking the absolute value of the integrand, and unimodality of the integrand, the negligible integral is $O(e^{-Cn^{1/5}})$. By multiplying the Gaussian integrand by the variable of integration to make it elementarily integrable, and after substituting in $\theta_0(n)$, we see that the tails of the completed Gaussian integral are $O(\frac{1}{\sqrt{n}})$. Lastly, we see that the sum of the terms of the Taylor Series expansion of the ln of the integrand are $O(n^{-1/5})$, which comes from the $O(n^{-1/5})$ asymptotic bound for the 3rd Taylor Series term, the fact that the length of the contour approaches 0, and the fact that we can bound the infinite polynomial multiplied to the 3rd Taylor Series term when the infinite polynomial is factored (save for the constant and quadratic terms). This can be utilized to show that the dominant integral is asymptotic to a Gaussian integral. By the complete Gaussian integral approximation, we get that the integral is asymptotic to $\sqrt{\frac{2\pi}{n}}$, and we can multiply this by the $\frac{e^n}{2\pi n^n}$ term to get the final asymptotic approximate mentioned above: $\frac{e^n}{n^n\sqrt{2\pi n}}$.

7. Multivariate Asymptotics

Multivariate Asymptotics is a very poweful tool in Analytic Combinatorics that is connected to probability distributions and can be utilized to prove important results such as the Central Limit Theorem, also having inspired much modern research in Analytic Combinatorics. Unfortunately, this section is mostly out of the scope of this paper, but due to its prevalence, we will provide a very brief and high level overview of the main ideas of multivariate asymptotics.

[FS09] The goal of multivariate asymptotics is to determine asymptotic formulas for the coefficients of multivariate generating functions, usually bivariate generating functions. The most simplest approach is to determine an asymptotic formula for one variable in terms of another variable, and determine an asymptotic for the asymptotic as the other variable approaches ∞ as well. If this is not possible, we often utilize the techniques developed in the earlier sections to, again, analyze the multivariate generating function with respect to 1 variable, assuming that the other variable is set constant at 1. Then, we examine how slight changes in the variable change the asymptotics for the coefficients as the other variable approaches infinity, and utilize this to determine asymptotics for both coefficients approaching infinity.

Multivariate asymptotics, also called ACSV (Analytic Combinatorics in Several Variables) is often utilized to study probability distributions of combinatorical objects, especially when one wants to find the probability of an object having a certain parameter value. Multivariate asymptotics can be utilized to prove the Central Limit Theorem, which is essential in statistics and an extremely important result (it states that all probability distributions determined by sums or averages of draws from another probability distribution approach the normal Gaussian distribution). This can be intuitively understood by mathematically showing that the convolution of 2 Gaussian functions is another Gaussian function, but it must be shown that all repeated convolutions of distributions eventually converge to a Gaussian function, which can be done, in a way, by ACSV. ASCV also provides methods of deriving other limit laws other than the CLT, which are deeply useful in a variety of different fields. Again, we will not go into depth about the specifics of ACSV, because it is outside the scope of the paper (ACSV is a large field itself), but we encourage the interested reader to do further reading into the references we have mentioned.

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