Exploring Permutahedra

A Journey into Combinatorial Geometry

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Introduction

The goal of this talk is to present the concept of the geometric structures known as “Permutahedra” and illustrate the rich combinatorial properties they hold.

We start by stating some basic information about them along with proof, for the purpose of developing some intuition. We then use these statements to prove more advanced statements.

The major results I aim on explaining in this talk are Rado’s theorem, and the combinatorial structure held by the number of facets of each dimension.

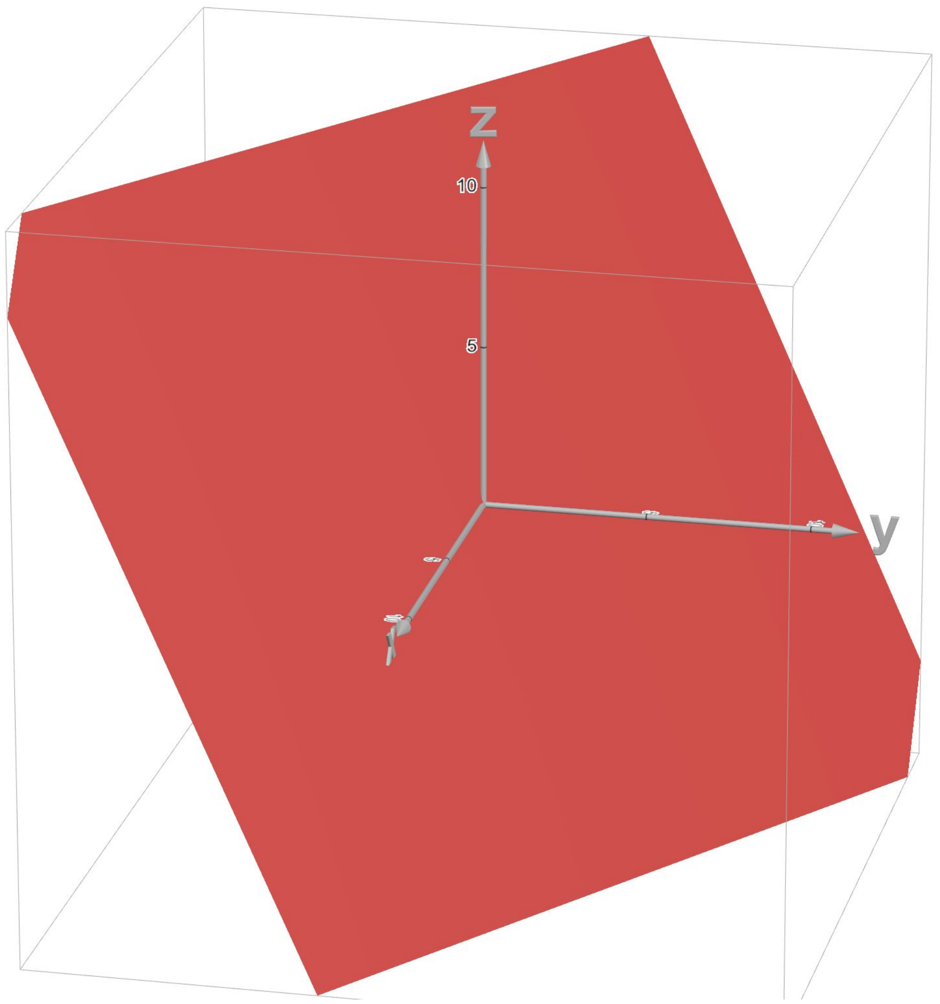
Preliminaries

Briefly explaining terms that will be used throughout the rest of the talk.

- R^n : The Euclidean space of dimension *n*. Points are represented by n-tuples.
- n-tuples: Ordered sets of real numbers with *n* elements. In this context, n-tuples represent the coordinates of a point in R^n
- Polytopes are higher dimensional analogues of polygons and polyhedra and are formally defined as the **convex hull** of a finite set of points known as its vertices.
- Affine spaces: Affine spaces are the higher dimensional analogues of planes. The dimension of an affine space, k, is strictly less than n. They can be defined by the following expression:

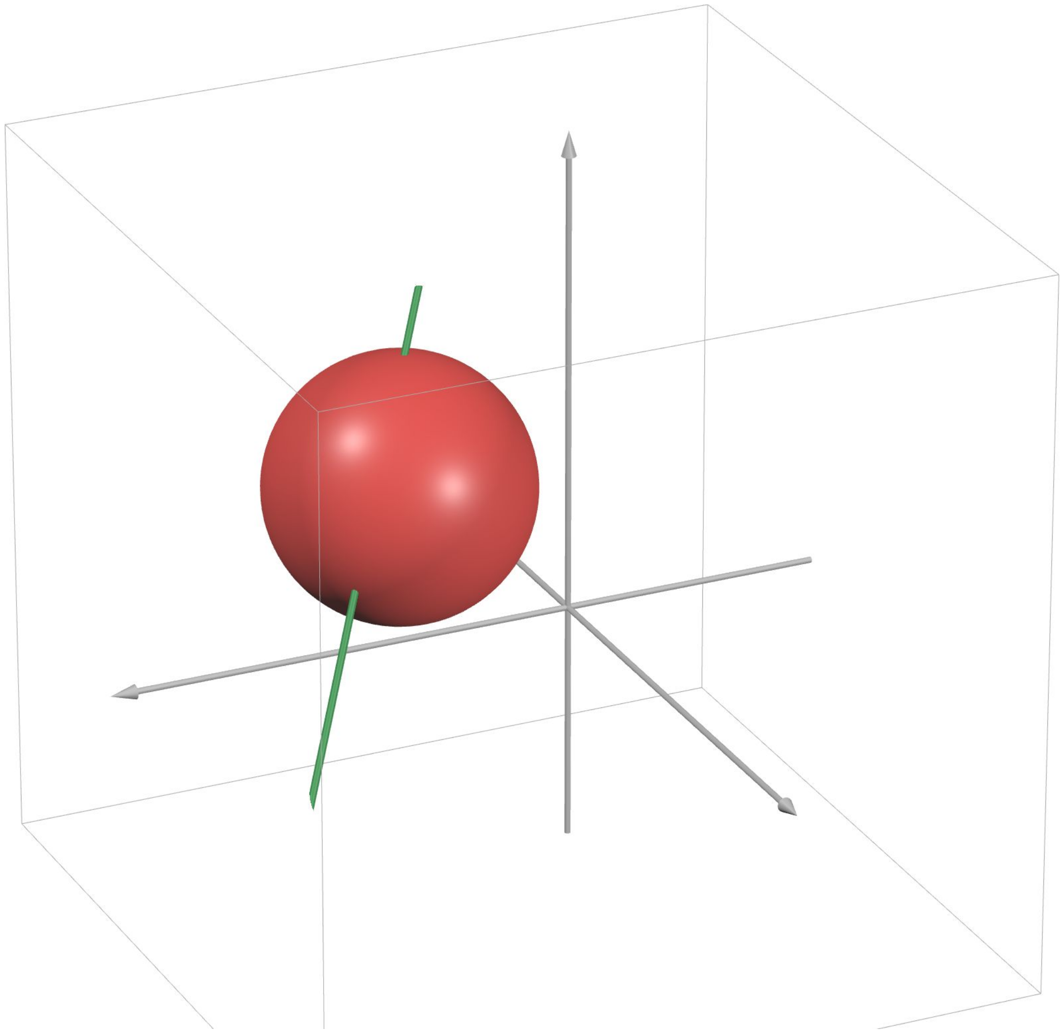
$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3} \cdots + c_{k+1}\mathbf{x_{k+1}} = 1$$

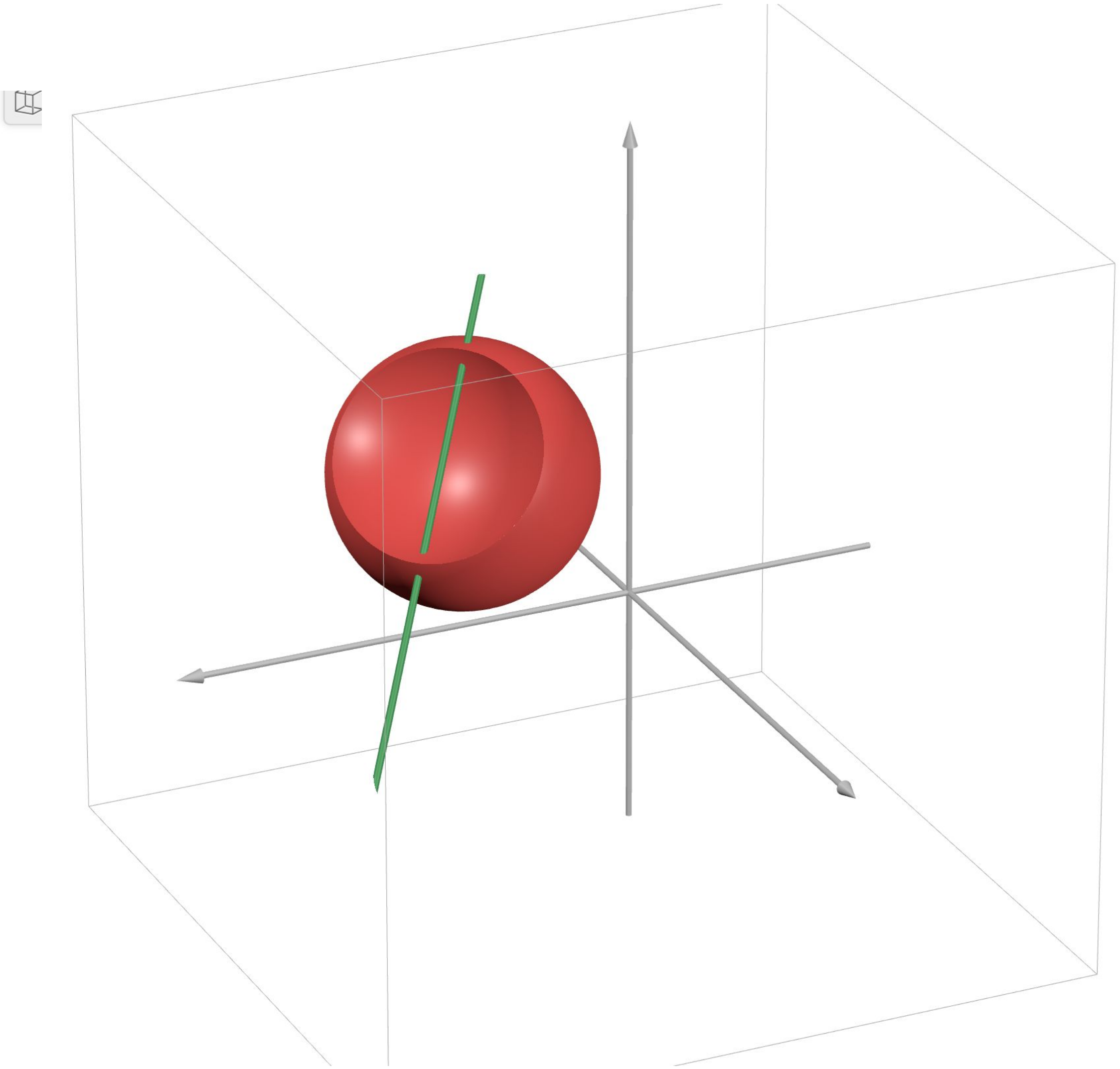
- Plane in R^3 defined by the equation 6x + 9y + 4z = 1



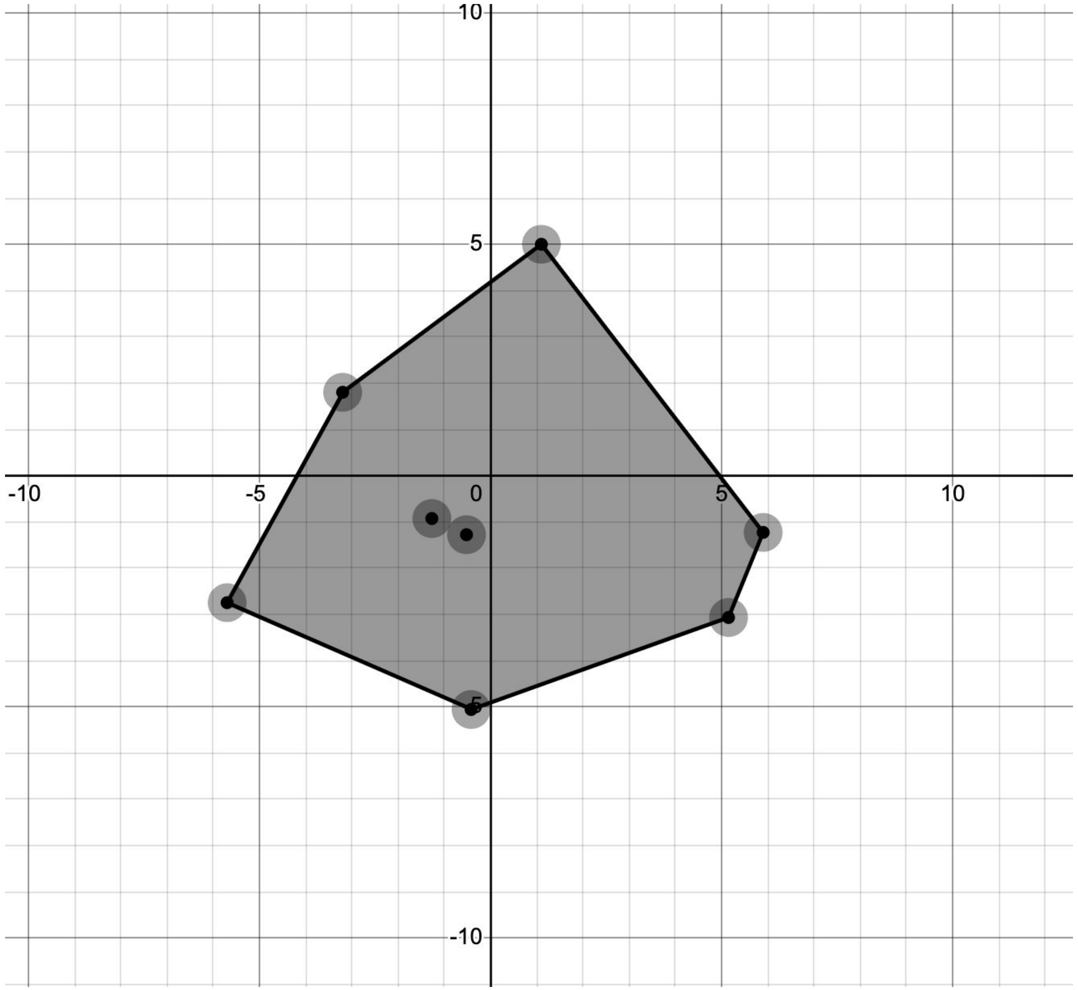
- Convexity: A set is convex if and only if it fully contains every line segment that has its endpoints in the set.

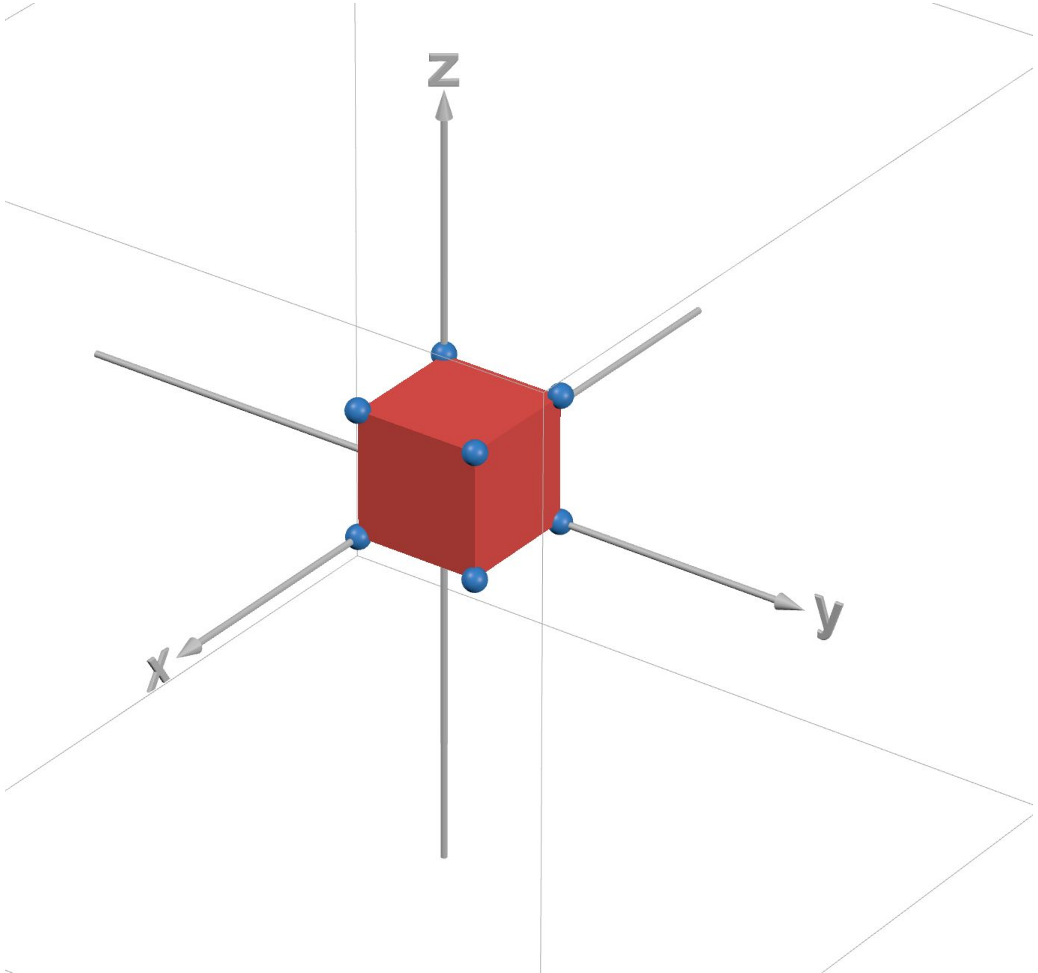
This definition is extremely important, as all polytopes (and by extension permutahedra) are, by definition, convex.





- Convex hull: The smallest convex set which contains a certain "root" set. It can also be defined as the intersection of all convex sets containing this root set.





- Facets: Facets are features of a polytope of dimension 1 through n-1. They are formally defined as

Permutahedra

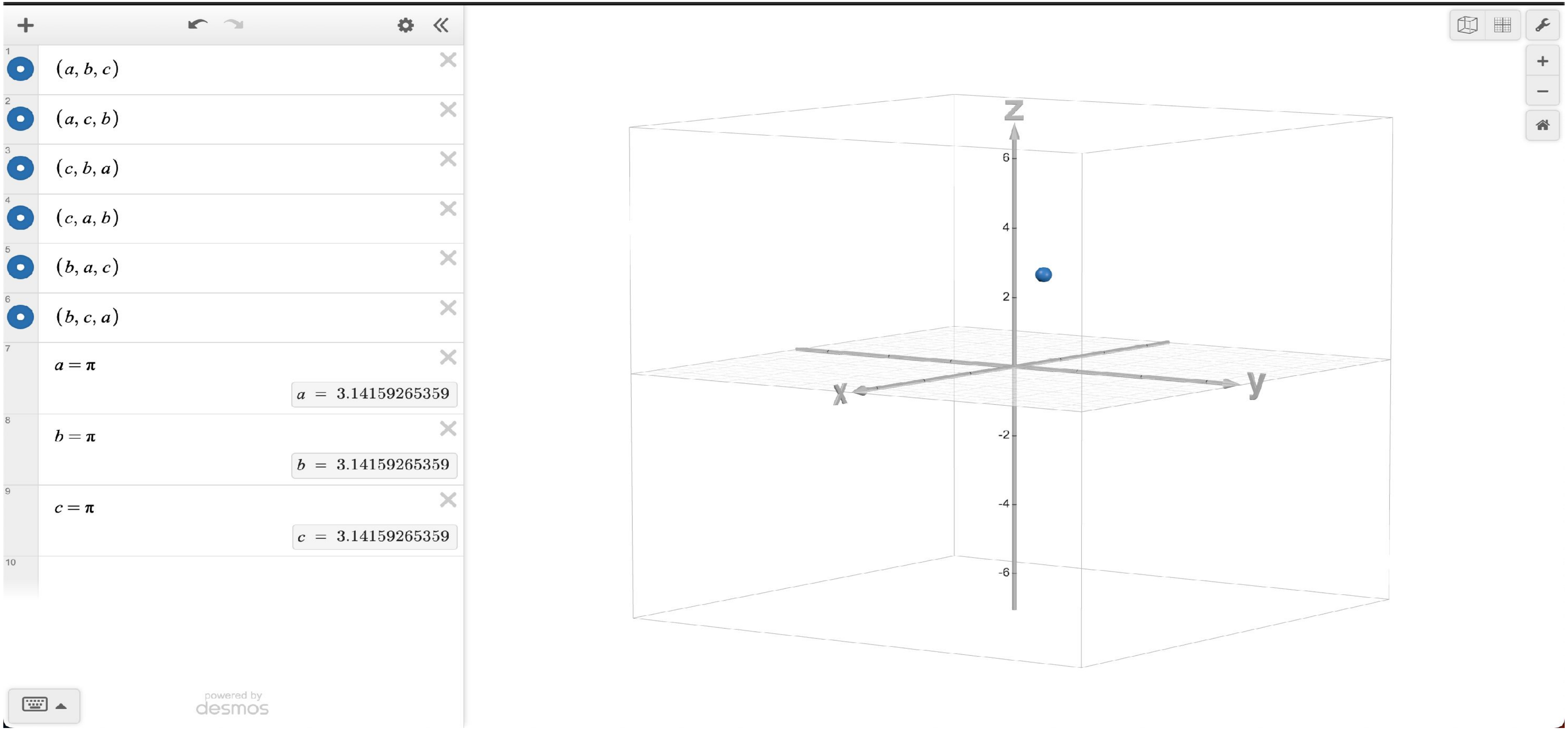
Introducing permutahedra, and some of their basic properties.

What are permutahedra?

- If one has an n-tuple, it represents a point in R^n. By applying the symmetric group to this n-tuple, we obtain all possible permutations of this n-tuple. This set of n-tuples represents n! points in R^n, which are the vertices of the permutahedron. The permutahedron itself is defined as the convex hull of these points, or equivalently the convex hull of its vertices.

Degenerate permutahedra

- For the purpose of this talk, we ignore degenerate permutahedra, since they do not hold the combinatorial properties that non-degenerate permutahedra do



(a, b, c)
(a, c, b)
(c, b, a)
(c, a, b)
(b, a, c)
(b, c, a)
a = π
a = 3.14159265359
b = π
b = 3.14159265359
c = π
c = 3.14159265359
powered by desmos
z
x
y
6
4
2
-2
-4
-6

1 $x_1 = (1, 2)$

Label

2 $x_2 = (2, 1)$

Label

3 $x + y = 3\{\max(x, y) < 2\}$

4 $y_1 = (7, 3)$

Label

5 $y_2 = (3, 7)$

Label

6 $x + y = 10\{\max(x, y) < 7\}$

7 (a,b)

Label

8 (b,a)

Label

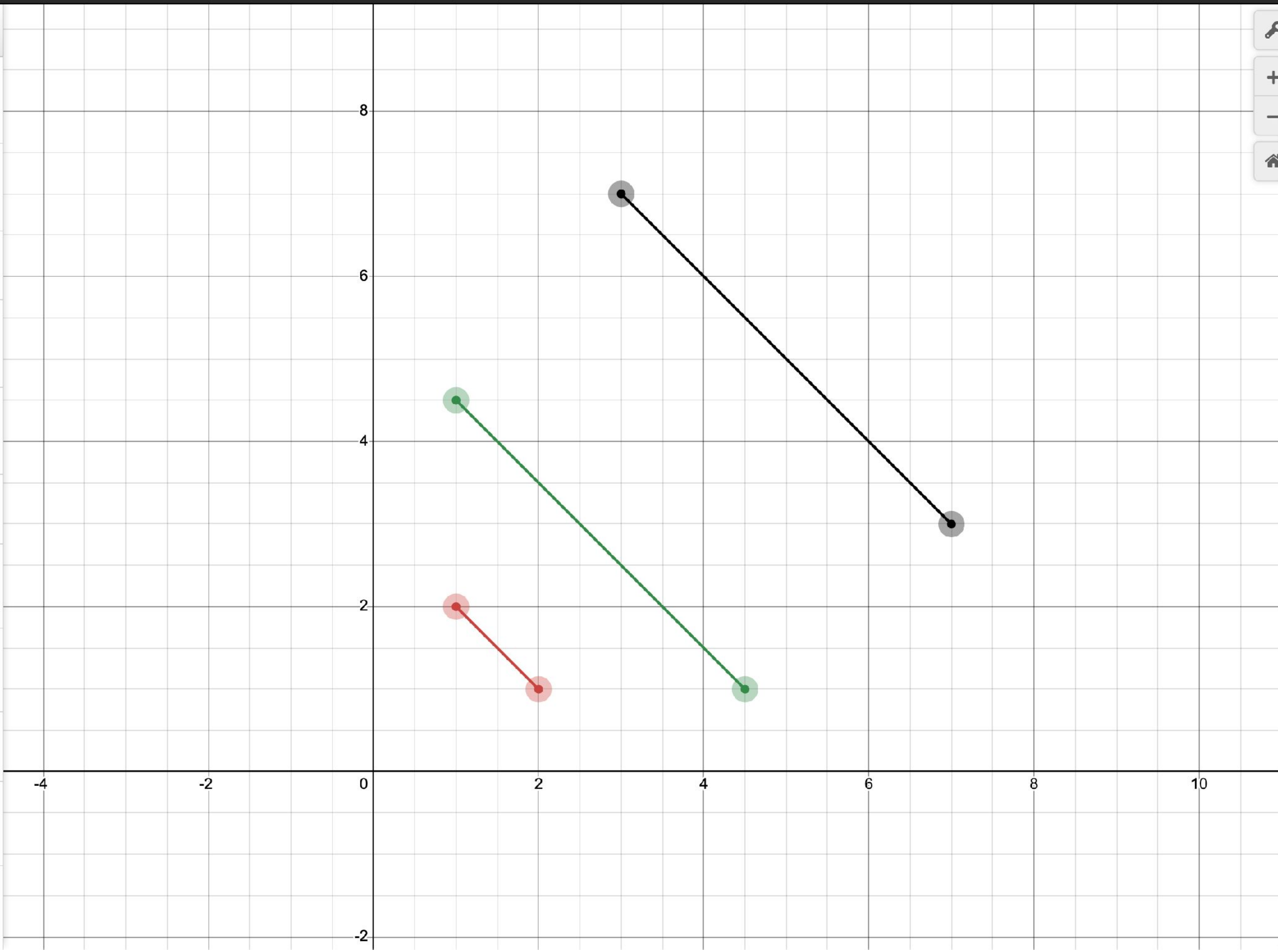
9 $x + y = a + b\{\max(x, y) < \max(a, b)\}$

10 $a = 4.5$

−10 10

11 $b = 1$

10



1 $mx + my + mz = 1$

2 (a, b, c)

3 (a, c, b)

4 (c, b, a)

5 (c, a, b)

6 (b, a, c)

7 (b, c, a)

8 $a = -0.35$

−5 5

9 $b = 4.77$

−5 5

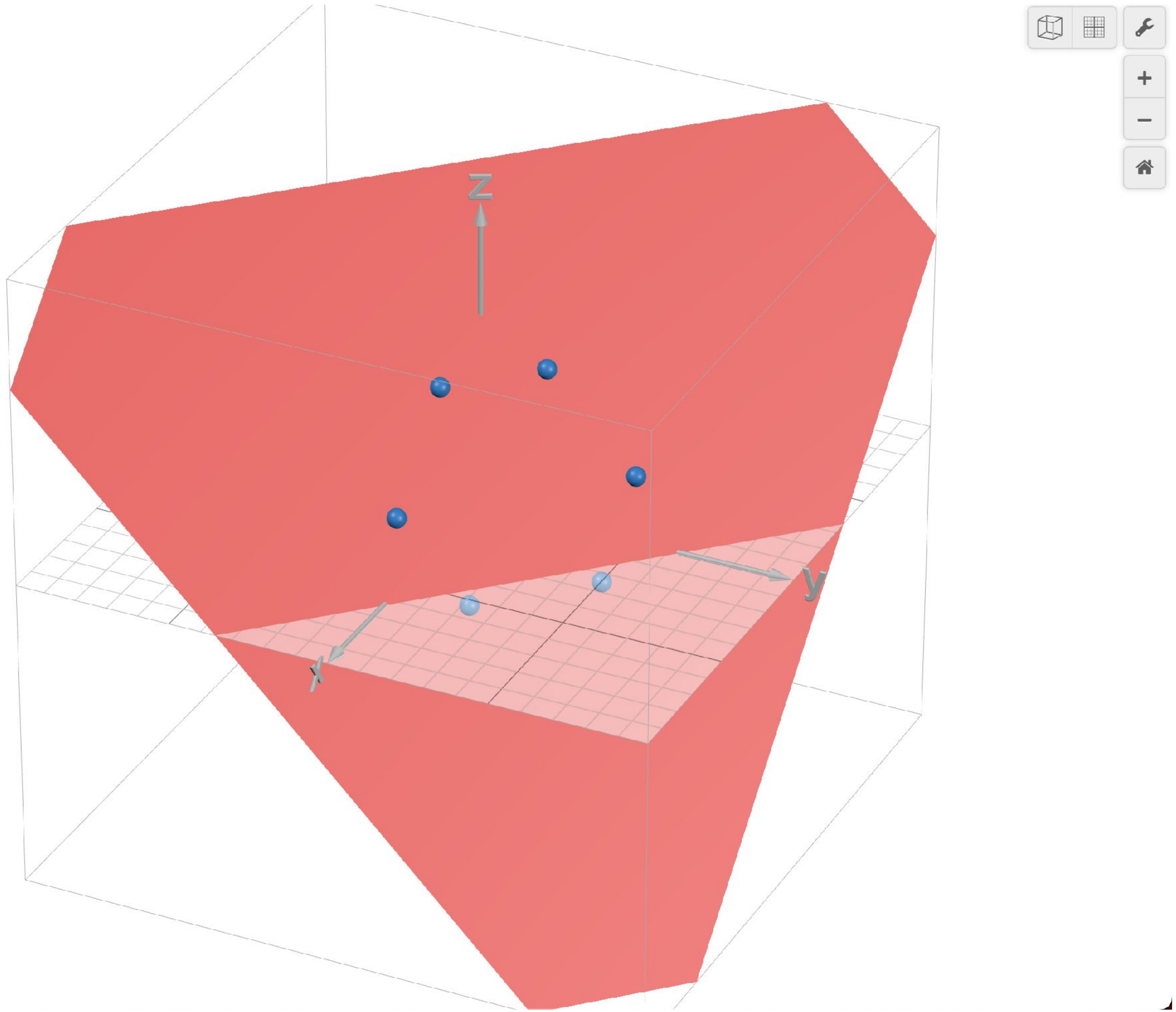
10 $c = 1.94$

−5 5

11 $m = \frac{1}{a + b + c}$

$m = 0.157232704403$

12



Permutahedra are of dimension n-1

- All permutahedra formed by an n-tuple are of dimension n-1. That is to say, there exists an affine space of dimension n-1 such that every point that is in the permutahedron is also in this affine space.

- Proof: Simply consider $c_1 = c_2 = c_3 = \dots c_n = \frac{1}{\Sigma N_i}$

By substituting these values into our equation of an affine plane, we can factor out the c_i terms, to obtain the equation of the plane as

$$c_i \sum_{i=1}^{n} x_i = 1$$

which trivially holds true for the vertices of the permutahedron

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3} \cdots + c_{k+1}\mathbf{x_{k+1}} = 1$$

The center of permutahedron

- The centre of every permutahedron lies at coordinates

$$(\alpha, \alpha, \alpha \ldots \alpha)$$

where $\alpha = \frac{\Sigma N_i}{n}$

Rigorous proof not provided.

Rado’s theorem

An alternate definition of permutahedra.

Rado’s theorem.

- Now that we have proven that every permutahedron is an n-1 dimensional object, we proceed with the statement of Rado’s theorem and its proof.

  Rado’s theorem proves that any point is contained by the convex hull of a permutahedron’s vertices if and only if a set of statements are true.

Condition 1: Only if

- First we prove that this statement is necessary for a point to lie inside a permutahedron. We already know $\frac{x_1}{m} + \frac{x_2}{m} + \cdots \frac{x_n}{m} = 1$ defines the plane of the permutahedron, where m is the sum of the n-tuple, or ΣN_i.

- Therefore, a point x can lie in the permutahedron only if $\sum_{i=1}^{n} N_i = \sum_{i=1}^{n} x_i$

Condition 2: If

- Now, we detail the second condition. This condition, combined with the first one, ensures that every point which fulfills these statements lies in the permutahedron.

- If i is a subset of {1, 2, 3 ... n}, and $N_1 > N_2 > N_3 \ldots > N_n$

  Then, $\Sigma_{i=j} y_i \leq \Sigma_{i=j} N_i$

The exact explanation is not provided for brevity, but this linear inequality works by using the coordinates of the vertices being represented by the n-tuple, and the definition of convexity. If the reader wishes to, they can attempt a proof by inducting on the number of dimensions or they can read my paper on June 14th.

Facets

Counting facets, and explaining their combinatorial properties.

Explanation

A quick way to understand facets of a polytope are the “lower dimensional” sides.

For example, a square has dimension 2, and its facets have dimension 0 and 1

A cube has dimension 3, and has facets of dimension 0, 1, 2.

This logic extends for polytopes in higher dimensions.

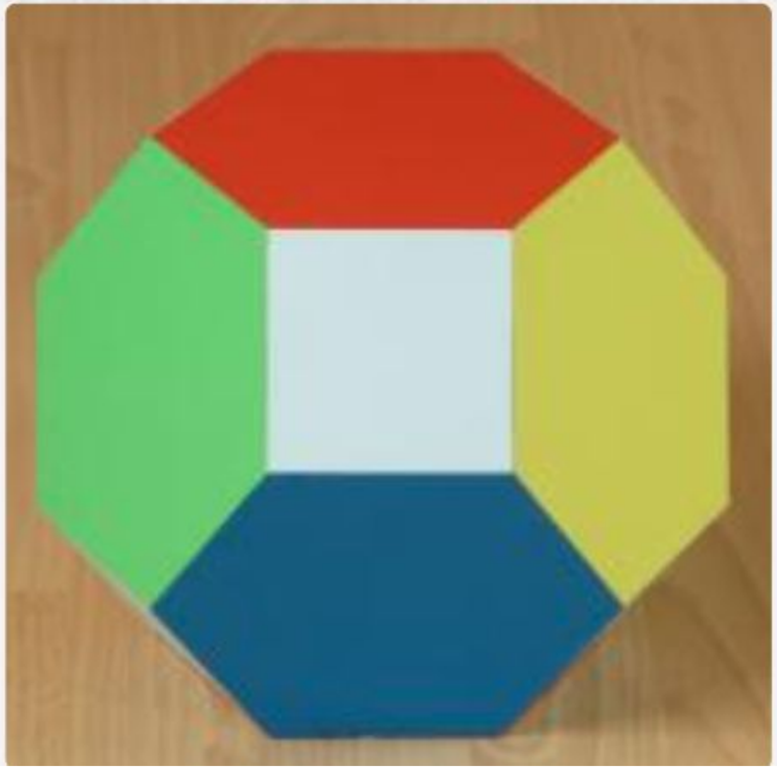
Counting facets

$$T(n-k) = k! \left\{ n \atop k \right\}$$

These numbers are the Stirling numbers of the second kind, and represent the number of ways to partition a set of n objects into k non-empty subsets

n \ k	1	2	3	4	5	
1	1					1
2	1	2				3
3	1	6	6			13
4	1	14	36	24		75
5	1	30	150	240	120	541

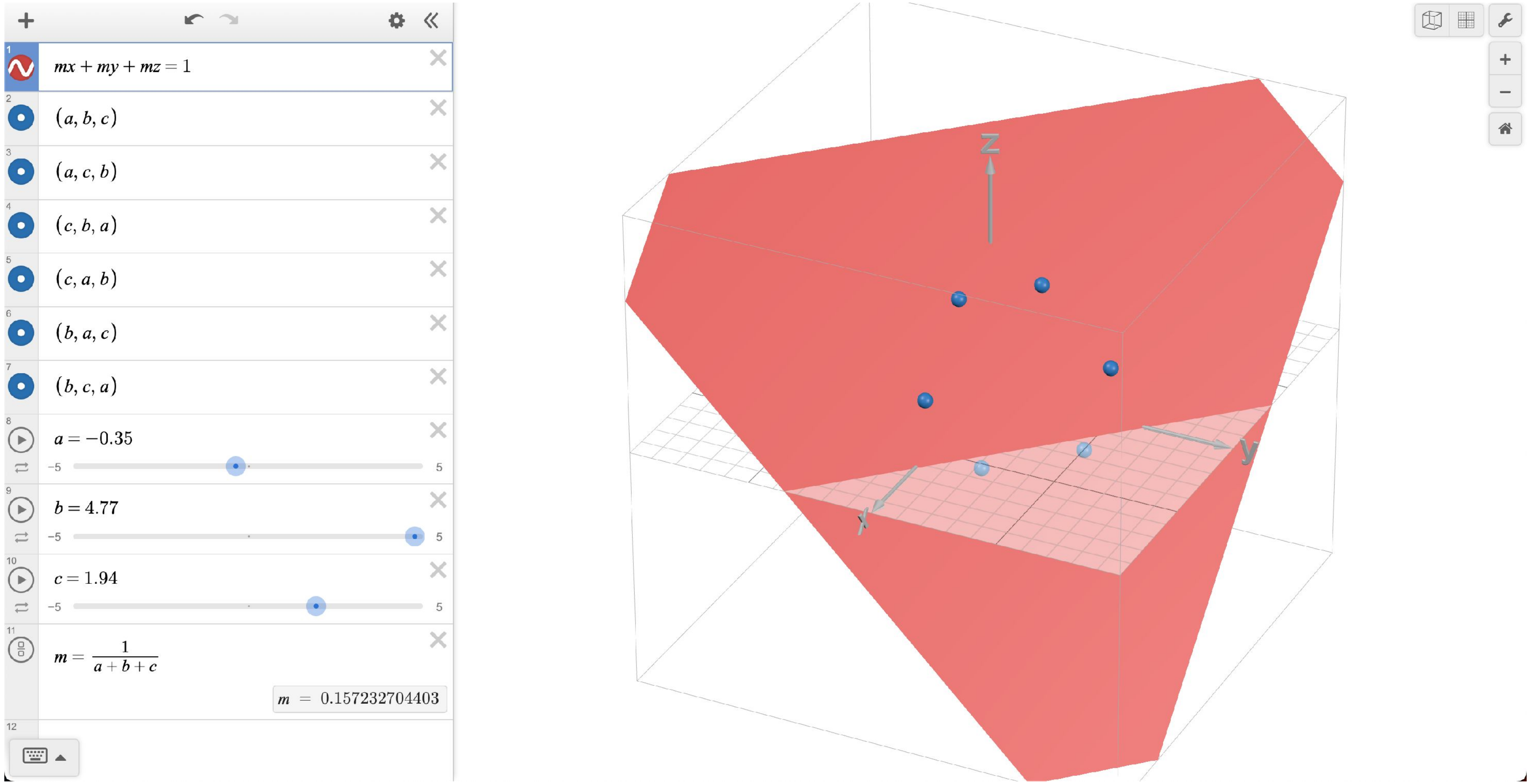
Quick applications.



Truncated Octahedron

Number of faces: 14
Number of edges: 36
Number of vertices: 24

Quick applications.



Some other properties

- The total number of facets (n-2 dimensional faces) is exactly $2^n - 2$, because each facet corresponds to the number of non-empty proper subsets of {1, 2, 3... n}

Fun fact!

- A not very relevant, but still quite interesting fact:

  If the members of the n-tuple form an arithmetic progression, the permutahedra is regular!

  Proof: Sides of polytopes are formed between vertices whose coordinates have exactly 1 pair of adjacent digits swapped. (adjacent when arranged in ascending order)

  Since all adjacent digits have a constant difference in permutahedra formed by APs, all edges have the same length

  note: the motivated reader could attempt this characterization of edges as a partial proof of the formula for number of facets.

Slide 7: Lectures on Polytopes, Günter M. Ziegler, page 16

Slide 8: https://www.desmos.com/3d/10zddkr3e8, https://www.desmos.com/3d/r2zlhc6gt9. (self made)

Slide 9: https://www.desmos.com/calculator/2bixtbca1d (credit to reddit user u/Knalb_a_la_Knalb), https://www.desmos.com/3d/snszty6jsk

Slide 14: https://www.desmos.com/3d/dulsd9rhf4

Slide 15: https://www.desmos.com/calculator/4dkbtr5hby

Slide 16: https://www.desmos.com/3d/81muaakm2p

Slide 27: https://www.polyhedra.net/en/result.php?type_fev=faces&operator=%3D&number=14&B1=Submit

https://polytope.miraheze.org/wiki/Permutohedra

Slide 28: https://www.desmos.com/3d/81muaakm2p

Thank You

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