Euler Circle Paper: Permutahedra

Moksh Nihalani

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1 Introduction

This paper is an expository paper, seeking to explain some basic properties and facts related to Permutahedra in an accessible manner to readers who have little to no background knowledge in this field of study. We also investigate some combinatorial properties they (Permutahedra) hold. The goal of the paper is to rigorously define permutahedra, detail some basic properties of Permutahedra that can be found through relatively simple methods, and then we use these results as a base upon which we prove more advanced properties.

The major results this paper aims to prove are Rado's theorem, the formula for the volume of a regular permutahedron, as well as the number of each type of **facets** a permutahedron has.

Contents

2 Preliminaries

2.1 Assumed knowledge

It is assumed the reader is familiar with the representation of points in $\mathbb{R}^n$ with respect to the standard basis, as well as intuition of basic properties of polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$, which are then extended to higher dimensions.

2.2 Definitions

Now, we define some basic terms that will come up frequently in the discussion of permutahedra.

1. Polytopes: Polytopes are higher-dimensional analogs of polygons, and are defined by the **convex hull** of a set of points in $\mathbb{R}^n$. Note that polytopes are necessarily, by definition, convex, unlike polygons.

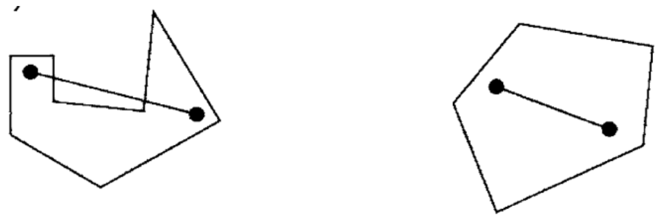
2. N-tuples: N-tuples are ordered sets of n numbers, where n is a positive integer. In this paper, n-tuples will be used to represent coordinates of points in $\mathbb{R}^n$, and thus its elements will be real numbers.

3. Affine spaces: Affine spaces are, informally, vector spaces without a 0 vector in the sense that they can give the coordinates of a point without reference to a 0 vector. Since this paper does not focus on affine spaces, we can think of them as a subspace of dimension $\mathbb{R}^k$ in $\mathbb{R}^n$ where $k < m$ combined with a translation. For visualisation purposes, consider the set of points in 3-dimensional space defined by the equation $x + y = z$ This is a affine subspace of dimension 2 in $\mathbb{R}^3$. If the affine subspace is of dimension $n - 1$, we specifically refer to it as a hyperplane

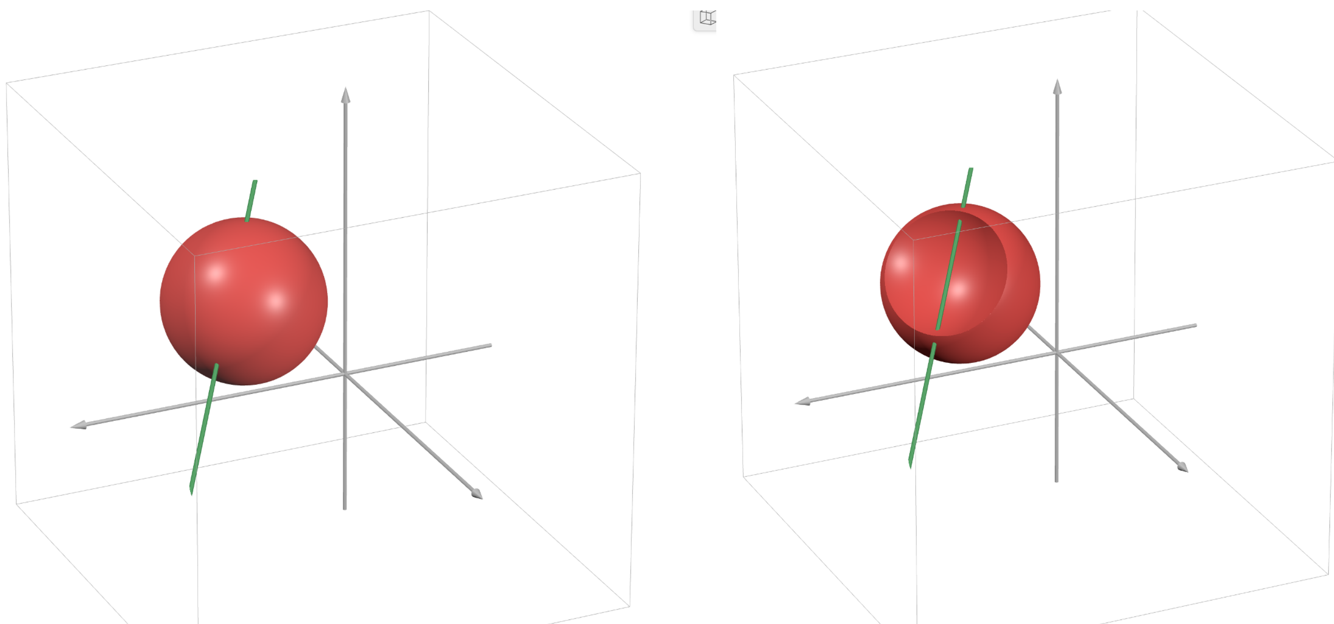
4. Facets: Facets are the n-dimensional sides of a polytope. Any polytope of dimension n will have sides of dimension $0, 1, 2, \ldots n - 2, n - 1$. For example, a 3 dimensional cube has 8 facets of dimension 0 (vertices), 12 facets of dimension 1 (edges), and 6 facets of dimension 2 (sides). Similarly, higher dimensional polytopes have facets of all dimensions strictly less than itself.

5. Convexity: A set of points S in $\mathbb{R}^n$ is said to be convex if $A, B \in S \longrightarrow A + t(B - A) \in S \forall t \in [0, 1]$ Intuitively, this statement states that for any two points A, B that are in set S, the line that connects the two points is also completely enclosed by set S.

Below is a quick visual to clarify the previous statement.



As we can see, the set on the left is not convex, since it doesnt contain the indicated line. The set on the right, however, is.



Similarly, the solid on the left is not convex, while the solid on the right is. Note that the lines have been produced indefinitely for viewing purposes, but an accurate test of convexity would be of line segments with endpoints in the solid. However, this would make the contrast less visible to readers.

6. Convex hull: A convex hull is the minimum convex set containing a set of "root" points. In the case of polytopes, the root points must be finite in number. An alternate way to think of the convex hull would be as the intersection of all sets containing the root set, since the intersection of convex sets preserves convexity.

Proof. Let A, B be sets in $\mathbb{R}^n$. We must prove that if A, B are convex, $A \cup B$ is convex If points $\alpha, \beta \in A \cup B \longrightarrow \alpha, \beta \in A, \alpha, \beta \in B$. Since $\alpha, \beta \in A$, $A + t(B - A) \in A \forall t \in [0, 1]$

and since $\alpha, \beta \in B, A + t(B - A) \in B \forall t \in [0, 1]$. Since $A + t(B - A) \in A, B \forall t \in [0, 1]$, $A + t(B - A) \in A \cup B \forall t \in [0, 1]$ $\square$

A more intuitive, yet extremely informal way, that personally helped me visualise the convex hull of a set of points is by imagining a large balloon that completely enclosed all the points, and then letting the air out. As the balloon shrunk, it would, beyond a point, be blocked from further shrinking by the points, and the surface of the balloon would represent the convex hull. Note that this explanation is highly informal, and only serves to potentially aid with building reader intuition.

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3 General discussion of permutahedra

3.1 Definition

Firstly, what are permutahedron? Permutahedra are defined as follows: If we have an n-tuple, the coordinates of the corresponding permutahedron are defined by the application of the symmetric group onto the coordinates. Note that the symmetric group of a set consists of all possible permutations of this set.

3.2 Basic properties

Trivially, we can see the a permutahedron of degree n will have $n!$ vertices, since a set of n elements has $n!$ permutations. For the sake of this paper, we ignore "degenerate" permutahedra, where multiple elements of the n-tuple are equal. This is due to the fact that they are degenerate results in them not possessing the same "neat" combinatorial properties that non-degenerate permutahedra possess.

The first interesting property of permutahedra is that n-dimensional permutahedra are in fact objects of dimension $n - 1$. That is to say, they are entirely contained by an affine space of dimension $n-1$. In fact, we can make a slightly stronger claim that this affine space which the permutahedron lies on will intersect all the axes at the same value.

Corollary 3.1. Permutahedra of degree n are objects of dimension $n - 1$, and the formula of the plane they lie on is of the form $\frac{\sum_{i=0}^{n} \mathbf{x_i}}{m} = 1$

Proof. Hyperplanes in $\mathbb{R}^n$ can be defined as equations of the form

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3} \cdots + c_n\mathbf{x_n} = 1$$

Or, equivalently, in the more compact form

$$\sum_{i=0}^{n} c_i\mathbf{x_i} = 1$$

. If we can find a set $C = \{c_1, c_2 \cdots c_n\}$ such that the above equation holds true for the coordinates of all the vertices of the permutahedron, we can prove it lies on a $n-1$ dimensional hyperplane and thus, is an $n-1$ dimensional object.

'If we set $c_1 = c_2 \ldots = c_n = \frac{1}{m}$, we can factor it out of the sum to obtain $\dfrac{\sum_{i=0}^{n} \mathbf{x_i}}{m} = 1$ and thus $\sum_{i=0}^{n} \mathbf{x_i} = m$ Now, regardless of the exact permutation of the n-tuple, the LHS is always constant. Therefore, we can simply compute the sum of the elements of the n-tuple, and set the coefficients to its reciprocal. Thus, the equation for the plane on which a permutahedron lies is given by $\sum x_i = \sum N_i$ where N_i is the "i"th member of the n-tuple when arranged in decreasing order. □

Note that the order does not matter in this case, but this convention is useful to adopt for other proofs within this topic.

In addition to that the centre of a permutahedron also has quite a neat expression.

Corollary 3.2. The centre of a permutahedron lies at the coordinates

$$x_n = \frac{\sum N_i}{n}$$

Proof. We define the "centre" as the point within the permutahedron which is equidistant from all the vertices. Consider the two vertices $A = (a, b, N_1, N_2 \cdots N_{n-2})$ and $B = (b, a, N_1, N_2 \cdots, N_{n-2})$

Clearly the centre $\mathrm{O} = c, d, x_1, x_2 \cdots, x_{n-2}$ must be equidistant from A and B.

By considering the Euclidean metric and ignoring terms that are the same for OA and OB, we obtain the equation $(a-c)^2 + (b-d)^2 = (a-d)^2 + (b-c)^2$. By expanding both brackets and subtracting the second equation from the first one, we obtain the equation $c^2 - d^2 + 2ad - 2ac = c^2 - d^2 + 2bd - 2bc$. By subtracting $(c^2 - d^2)$ from both sides and factorising, we obtain $2a(d-c) = 2b(d-c)$. Now, either $a = b$, or $d - r = 0$. Since a and b are members of our n-tuple, any arbitrary permutahedron can be constructed such that $a \neq b$. Thus, we must conclude that $d - r = 0, d = r$. Since this exact process can be repeated by swapping two elements of any x_i, we conclude that any two coordinates must be equal, thus every two coordinates must be equal, and the centre has the equation $x_1 = x_2 = \cdots = x_n = k$. Since we know the centre of the permutahedron must lie on the plane of the permutahedron itself,

$$k + k \cdots + k = nk = \sum N_i \Longrightarrow k = \frac{\sum N_i}{n}$$

Alternatively, this value k can be thought of as the mean of the n-tuple.

□

3.3 Alternate definition

Although it appears relatively self explanatory, it is important to note that permutahedra are defined as the convex hull of the set of all permutations of their root n-tuple. Although this definition follows directly from their definition, and the definition of polytopes, it's extremely important to conceptualise.

3.4 Rado's theorem

An extremely important result in the discussion of polyhedra is Rado's theorem, which characterises whether any given point lies within a certain permutahedron. We now explain this characterisation, as well as provide a proof for Rado's theorem.

Rado's theorem states that a certain point lies inside a permutahedron if and only if two criteria are fulfilled. Let y_i denote the x_i coordinate of the point we are testing for, and recall the N_i convention we adopted earlier. The two criteria are as follows; firstly,

$$\sum y_i = \sum N_i$$

and secondly,

$$\sum_{i \in S} y_i \leq \sum_{i=1}^{|S|} N_i$$

The first statement simply states that the sum of the coordinates of the point must be equal to the sum of the n-tuple, which is a result we have already covered in sufficient detail in 3.1. This statement simply states that a point must lie on the plane that defines a permutahedron to be inside a permutahedron, which seems quite intuitively true. For example, if one has a certain solid in $\mathbb{R}^3$, for a point to be inside that solid, it must first be in 3 dimensions. Any point with non-zero vector components of higher dimensions will obviously, no matter what, not lie in this solid.

However note that this is merely an *only if* condition. That is to say- it's a *check* for whether a point is in a permutahedron. We know that any point that doesnt satisfy this check is not within the permutahedron. However, it doesn't ensure that every point that satisfies this test *does* lie within the permutahedron. That's where the second part of Rado's theorem comes into play

The second statement, although appearing somewhat difficult to digest, can be equivalently expressed in the following manner: For any number "S" less than n, the sum of the S greatest members of the n-tuple must be greater than or equal to the sum of *any S* coordinates of the point. Although this definition may at first seem quite arbitrarily chosen, the proof can be explained with the techniques and ideas introduced so far, albeit with some difficulty.

If we consider two points A & B that satisfy this second condition, it is (with some consideration) easy to see that $tA + t'B$ also satisfies these conditions, where t, t' are real numbers that sum to 1. Therefore if A, B lie within the permutahedron, every point between them also lies within the permutahedron, since we have simply restated the definition of convexity, and the permutahedron is merely the convex hull of its vertices. Now, we can simply induct on the number of dimensions to prove that every point satisfying this linear inequality must lie within the permutahedron, completing the *if* condition of the proof, proving Rado's Theorem in both directions.

If a point (y_1, y_2) lies on this line segment, $y_1 < x_1$ and $y_2 > x_2$

4 Properties of Regular permutahedra

4.1 Definition

So what are regular permutahedra? Like regular polygons, regular permutahedra are defined as permutahedra with all side lengths equal to each other.

4.2 Elaborating further

Recall that sides of a permutahedra are between vertices that have identical coordinates, except with the relative positions of two vertices, N_k and N_{k+1}, swapped (recall our convention of numbering elements of the n-tuple based on ordering). Note that this implies the pair of vertices swapped must be consecutive when arranged in decreasing order.

4.3 Properties of n-tuple

One interesting aspect of regular polytopes, is that the property of regularity allows us to infer certain information about the root n-tuple. For all sides to be equal, the distance between the points with N_k and N_{k+1} swapped must be the same. Since all the coordinates except two are the same, all the terms in the Euclidean metric cancel out, and we're left with the expression $\sqrt{2}\,(N_k - N_{k+1})$ For this expression to be constant for all choices of k, it's obvious that the difference of successive terms of the n-tuple must be constant, and the members of the n-tuple must form an arithmetic progression.

4.4 Non-applicability to higher dimensional facets

Unfortunately, this notion of regularity can't be applied to the higher dimensional facets of the permutahedra. That is to say, the volumes/hypervolumes of 2-D, 3-D$\cdots$ facets are not all equal

The proof of this is simple: Note that a non-degenerated permutahedron generated by a 4-tuple (also known as $\mathbb{P}_4$) is in fact a truncated octahedron - a 3-D solid with 6 square and 8 hexagonal faces. Since the square faces and hexagonal faces have the same side length, they trivially dont have the same area . To extend this idea to higher-dimensional facets, one can simply notice that permutahedra have more than one type of polytope as higher-dimensional facets, indicating that they have different volumes since they have the same sidelength.

5 Combinatorial properties of permutahedra

5.1 Counting facets

This section is, in my opinion, the most interesting aspect (so far) of permutahedra, and it characterises the exact combinatorial structure given by a permutahedron's facets. Similarly to how we characterised the edges of a permutahedron as necessarily having vertices differing by only one swap, faces of dimension $n-k$ can be characterised as being in **bijection** with the subdivision of $\{1, 2, 3 \cdots n\}$ into k disjoint, non-empty blocks. The number of ways to perform this subdivision of the first n numbers into k disjoint sets is given by the Stirling numbers of type 2, denoted by $\left\{ n \atop k \right\}$. However, note that these Stirling numbers only relate to the exact subsets/blocks made, whereas the faces of a permutahedron are given by all possible permutations as well. One can observe that each partition of the first n numbers into k subsets, will also have $k!$ permutations (since all the subsets are distinct). Therefore, we notice that the number of facets of dimension $n-k$ is given by the extremely compact equation

$$T(n-k) = k! \left\{ n \atop k \right\}$$

5.2 Volume

An extremely interesting fact about permutahedra is that if we project the regular permutahedron given by $P(0, 1, \cdots n-1)$ onto dimension $n-1$ (by replacing the coordinate of any one of the axes with 0 in all the vertices) the volume of this projected solid will always be n^{n-2}. Although a proof is not provided, since it is outside the scope of explanation of this paper, it's a fact that some reader might find particularly interesting, and could potentially incite the desire for further research on this topic.

6 Acknowledgements

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7 Referencing

Throughout the process of researching for, and writing, this paper, the main sources I referred to were: "Lectures on Polytopes" by Günter M. Ziegler, "Permutahedra, Associahedra, and Beyond" by Alexander Postnikov, and Dora Woodruff's exposition of the previous paper.

8 Further reading material

For any readers who are particularly fascinated in the field of study I briefly detailed in this paper, there were a large number of other topics of research I did not cover in this paper for the sake of brevity. However, the particularly dedicated reader might find it fascinating to further research topics such as the Euler characteristic and how it relates to polytopes & permutahedra, the exact proof of the standard permutahedron volume formula, which involves graph theory, a more formal definition of features using the minimisation of the dot product, and perhaps some research into degenerate polytopes as well.