Set Theory

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Introduction

Definition of a Set

Set: A set is a collection of distinct objects, considered as an object in its own right. The objects within a set are called elements or members of the set.
i.g. $\{2, 3, 5, 7\}$.
Subset: A subset is a set whose elements are all contained within another set. If every element of set A is also an element of set B, then A is called a subset of B, denoted as $A \subseteq B$
i.g. $\{3, 5, 7\}$.
Empty set: The empty set, denoted by $\emptyset$ or sometimes by $\{\}$, is a set that contains no elements at all.
Power set: The power set of a set S is the set of all subsets of S, including the empty set $\emptyset$ and S itself. If S has n elements, then the power set of S has 2^n elements. For example, if $S = \{a, b\}$, then the power set $P(S)$ is $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
i.g. $\mathcal{P}(\{2, 3, 5, 7\}) =$
$\{\emptyset, \{2\}, \{3\}, \{5\}, \{7\}, \{2, 3\}, \{2, 5\}, \{2, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{2, 3, 5\}, \{2, 3$

Russel's Paradox

Russel's Paradox

We consider the set, R, of all the sets which don't contain themselves. $R = \{x | x \notin x\}$. Does this set contain itself? The contradiction is that R can not contain and not contain itself at the same time.

Zermelo-Fraenkel Set Theory (ZFC)

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Axiom of Extensionality:
$\forall A, B : (A = B) \iff (\forall x(x \in A \leftrightarrow x \in B)) \wedge (\forall y(y \in bB \leftrightarrow y \in A))$.
Axiom of Regularity: $\exists B$ such that $\forall x$, if x exists, then $x \notin B$.
Axiom of Pairing: $\forall x, y \; \exists S$ such that $S = \{x, y\}$
Axiom of Union:
$\forall x, \exists S$ such that $\forall y(\exists z \in x : y \in z \Rightarrow y \in S) \wedge (y \in S \Rightarrow \exists z \in x : y \in z)$
Axiom of Power Set: $\forall A \, \exists B \, \forall S \, (S \in B \iff S \subseteq A)$

Zermelo-Fraenkel Set Theory (ZFC)

Axiom Schema of Separation:
$\forall A \, \forall \phi \, \exists B \, \forall x (x \in B \iff (x \in A \land \phi(x)))$
Axiom of Replacement:
$\forall A \, \forall \phi \, (\forall x \in A \, \exists ! y \, \phi(x, y)) \Rightarrow \exists B \, \forall y \, (y \in B \iff \exists x \in A \, \phi(x, y))$
Axiom of Infinity: $\exists \, \text{set} \, \mathbb{N} \, (\emptyset \in \mathbb{N} \land \forall x (x \in \mathbb{N} \Rightarrow x \cup \{x\} \in \mathbb{N}))$
Axiom of Choice:
$\forall x \, (\forall S \in x \, (S \neq \emptyset \land \forall A, B \in S \, (A \neq B \Rightarrow A \cap B = \emptyset)) \Rightarrow \exists c \, \forall S \in x \, (c(S) \in$

Axiom of Choice

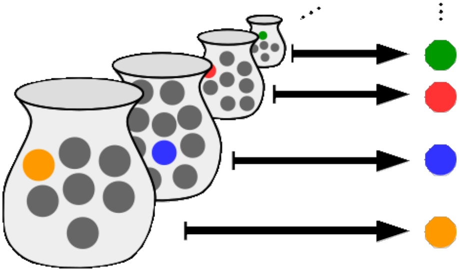

Figure 1: Axiom of Choice

Orderings

Partial Orderings

A partial ordering is a relation R meaning:

- R is a transitive relation: xRy and $yRz \implies xRz$
  i.e., $x < y$ and $y < z \implies x < z$
- R is irreflexive: $\neg(xRx)$
  i.e., $\neg(x < x)$

Linear Orderings

If a partial ordering relates every ordering in a set it is called a linear ordering.
A linear ordering, R, in a set A is defined as:

- Is a partial ordering
- Satisfies trichotomy on A
  For any x and y in A, exactly one of the following statements is true: xRy, yRx, or $x = y$

This condition makes sure that for every pair of elements in A, one of those elements is less than the other, or else, they are the same element.

Well Orderings

A well ordering on a set A is a linear ordering with the property that every non-empty subset of A has a least element.
i.e. The ordering $< \mathbb{R}$ is a linear ordering but not a well ordering because there is not one real number that is less than every other real number. On the other hand, $< \mathbb{N}$ is a well ordering because every possible subset of the positive integers has some minimal value. Ordinal numbers are well-ordered.

Ordinals

Ordinals Defintions

An ordinal is a transitive set well-ordered under the relation $x < y$ if $x \in y$.
Let α be an ordinal, and define the successor of α, denoted by $\alpha + 1$, to be $\{\alpha\} \cup \alpha$.
A limit ordinal is an ordinal α such that there does not exist an ordinal β with $\beta + 1 = \alpha$.
Let ω be the smallest set of ordinals closed under successor such that $\emptyset \in \omega$.
ω is a limit ordinal.

Induction

Induction

Induction usually consists of two steps:
Base Case: Prove that the statement holds true for the simplest case, typically $n = 1$ or $n = 0$.
Inductive Step: Assume that the statement is true for an arbitrary but fixed $n = k$ (k is an integer). Use this assumption to prove that the statement is also true for $n = k + 1$.
i.e. The sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Transfinite Induction

Base Case: The base case of transfinite induction is the same as regular induction.
Successor Case: Assuming the statement is true for all previous elements up to α, prove it for the next element, $\alpha + 1$.
Limit Case (Transfinite Step): If α is a limit ordinal (a step beyond which there are no immediate successors), prove that the statement holds for all previous ordinals less than α.
i.e. Consider proving a property $P(\alpha)$ for all ordinal numbers α:
Base Case: Prove $P(0)$.
Successor Case: Assume $P(\alpha)$ holds, prove $P(\alpha + 1)$.
Limit Case: If α is a limit ordinal, assume $P(\beta)$ holds for all $\beta < \alpha$, prove $P(\alpha)$.

Recursion

Recursion

Base Case: This is the simplest form of the problem that can be directly solved without further recursion.
Recursive Case: This defines how the function behaves for larger instances of the problem, typically by reducing the problem size and applying the same function to the reduced problem.
i.e. Factorial numbers use recursion

Transfinite Recursion

Transfinite recursion extends recursion to more than just the natural numbers. It is used on ordinals.
An example of transfinite recursion is ordinal arithmetic.

Ordinal Arithmetic

Addition

The addition of ordinals α and β, denoted $\alpha + \beta$, is defined based on the well-ordered nature of ordinals:

1. **Successor Ordinals ($\beta = \gamma + 1$):**

$$\alpha + (\gamma + 1) = (\alpha + \gamma) + 1$$

   This means adding β to α results in α incremented by 1 after adding γ.

2. **Limit Ordinals:**

$$\alpha + \beta = \sup\{\alpha + \gamma \mid \gamma < \beta\}$$

   For a limit ordinal β, $\alpha + \beta$ represents the supremum of $\alpha + \gamma$ for all $\gamma < \beta$.

Multiplication

The multiplication of ordinals α and β, denoted $\alpha \cdot \beta$, is defined recursively:

1. **Base Case:**

$$\alpha \cdot 0 = 0$$

Here, 0 represents the smallest ordinal.

2. **Successor Ordinals ($\beta = \gamma + 1$):**

$$\alpha \cdot (\gamma + 1) = (\alpha \cdot \gamma) + \alpha$$

α multiplied by the successor of β is α multiplied by β, then incremented by α itself.

3. **Limit Ordinals:**

$$\alpha \cdot \beta = \sup_{\gamma < \beta} (\alpha \cdot \gamma)$$

For limit ordinal β, $\alpha \cdot \beta$ is the supremum of $\alpha \cdot \gamma$ for all $\gamma < \beta$.

Exponentiation

Exponentiation α^β for ordinals is defined recursively:

1. **Base Case:**

$$\alpha^0 = 1$$

   Here, 0 represents the smallest ordinal.

2. **Successor Ordinals ($\beta = \gamma + 1$):**

$$\alpha^{\gamma+1} = \alpha^\gamma \cdot \alpha$$

   α raised to the successor of β is α raised to β, then multiplied by α itself.

3. **Limit Ordinals:**

$$\alpha^\beta = \sup_{\gamma<\beta} \alpha^\gamma$$

   For limit ordinal β, α^β is the supremum of α^γ for all $\gamma < \beta$.

Cantor's Normal Form Theorem

Cantor's Normal Form Theorem

Cantor's Normal Form Theorem states that every ordinal number can be uniquely expressed in a specific canonical form using a sum of ordinal powers of ω.

1. **Base Case:** Every finite ordinal n can be represented as ω^n.
2. **Inductive Step:** Assume every ordinal less than α can be uniquely represented as $\omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k}$, where $\beta_1 < \beta_2 < \cdots < \beta_k < \alpha$.
3. **Constructing Normal Form:**
   - For any ordinal α, find the largest $\beta < \alpha$ such that $\alpha = \beta + \gamma$, where $\gamma < \beta$.
   - Express γ in its normal form $\omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_m}$, where $\gamma_1 < \gamma_2 < \cdots < \gamma_m < \beta$.
4. **Canonical Form:**
   - Combine β and γ's normal form to get $\alpha = \omega^{\beta} + \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_m}$.
5. **Uniqueness:**
   - Show that this representation is unique by demonstrating that any two representations for α must be identical, ensuring β and γ_i's are uniquely determined.