# Set Theory

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## 1 Abstract

This paper goes into basic topics on Set Theory of Russel's Paradox, Zermelo-Fraenkel set theory, the Axiom of Choice, and Zorn's Lemma. Then it talks about sets, types of orderings, and ordinals. Finally, it discusses Cantor's Normal Form Theorem.

## 2 Introduction

A set is a group of members of elements. Set $A$ is said to be a subset of Set $B$ if all the elements of Set $A$ are also present in Set $B$. A power set of $A$ is defined as the set of all subsets of $A$. Cardinality is the number of elements in a given set. For any set $A$ with cardinality $|A|$, $P(A)$ has the cardinality of $2^{|A|}$.

Let's say that the set $V_0$ is a finite set. Then, $V_1$ will be all of the elements of $V_0 \cup P(V_0)$. For every $n$, $V_{n+1} = V_n \cup P(V_n)$ . No matter, how large of a number $n$ is, the set $V_n$, will still be a finite set because it is a group of finite elements and all of the combinations that can be made with those elements.

Lets say $V_w = V_0 \cup V_1 \cup V_2 \cup V_3$...Then, $V_w$ is equal to $V_\infty$ will be an infinite set because it $V_w + 1 = V_w \cup P(V_w)$. In this way, you can define $V_w, V_{w+1}, V_{w+2}$.... These are called ordinals. Ordinals define a different size of infinity.

For $V_n, V_{n+1}, V_{n+2}$... every set is made up of a combinations of finite sets, so all of them are the same order of size, which is finite. Since $V_w$ is already an infinite set, $V_{w+1} = \infty + 2^\infty$ and $V_{w+2} = \infty + 2^\infty + 2^{\infty + 2^\infty}$. This means that each $V_{w+1}$ is a bigger size of infiniti than $V_w$.

# 3 Russel's Paradox

We consider the set, R, of all the sets which don't contain themselves. $R = \{x|x \notin x\}$. Does this set contain itself? The contradiction is that $R$ can not contain and not contain itself at the same time. Zermelo-Fraenkel set theory (ZFC) resolves such paradoxes by rigorously defining what a set is.

# 4 Zermelo-Fraenkel Set Theory (ZFC)

**4.1. Axiom of Extensionality**: Two sets are equal if and only if they have the same elements.

**4.2. Axiom of Regularity**: Every non-empty set $x$ contains an element $y$ that is disjoint from $x$.

**4.3. Axiom of Pairing**: For any sets $x$ and $y$, there exists a set $\{x, y\}$ that contains exactly $x$ and $y$.

**4.4. Axiom of Union**: For any set, $x$, there exists a set with exactly those elements which belong to some element of $x$.

**4.5. Axiom of Power Set**: For any set $x$, there exists a set $P(x)$ that contains all subsets of $x$.

**4.6. Axiom Schema of Separation**: For any formula $p$ with parameters, if $x$ is a set, then $\{x \in A|p(x)\}$ is also a set.

**4.7. Axiom of Replacement**: For any set, $x$, and any function, $f$, there is a set that consists of exactly those elements related to elements in $x$.

**4.8. Axiom of Infinity**: There exists an infinite set.

**4.9. Axiom of Choice**: For any set $x$ of pairwise disjoint non-empty sets, there exists a set $c$ (a choice function) that chooses exactly one element from each set in $x$.

# 5 Orderings

Orderings are ways to take elements in a set and put them in some sort of order, where elements go before or after each other. One example of an ordering is the ordering less than on the real numbers. The set $\mathbb{R}$ doesn't have an order on its own but if we define what it means for one number to be less than another number we have a strictly ordered line of all the real numbers. All orderings are relations. An ordering is a set of ordered pairs each of which orders one element before the other. Less than isn't the only possible ordering. For instance, if we have a set of 50 people, we can define a relation, $T :< x, y >\in T$ if and only if $x$ is taller than $y$.

**1. Partial Orderings** A partial ordering is a relation $R$ meaning:

- $R$ is a transitive relation: $xRy$ and $yRz \implies xRz$

  i.e. $x < y$ and $y < z \implies x < z$

- $R$ is irreflexive: $x \not R x$

  i.e. $x \not< x$

**2. Linear Orderings** If a partial ordering relates every ordering in a set it is called a linear ordering.

A linear ordering, $R$, in a set $A$ is defined as:

- Is a partial ordering

- Satisfies trichotomy on $A$

  For any $x$ and $y$ in $A$, exactly one of the following statements is true: $xRy$, $yRx$, or $x = y$

This condition makes sure that for every pair of elements in $A$, one of those elements is less than the other, or else, they are the same element.

**3. Well Orderings** A well ordering on a set $A$ is a linear ordering with the property that every non-empty subset of $A$ has a least element.

i.e. The ordering $< \mathbb{R}$ is a linear ordering but not a well ordering because there is not one real number that is less than every other real number. On the other hand, $< \mathbb{Z}_{>0}$ is a well ordering because every possible subset of the positive integers has some minimal value. Ordinal numbers are well-ordered.

# 6 Axiom of Choice

Let's say there are 5 sets and in each set, there are 5 indistinguishable elements. We can choose one element from each set and put it in a new set because of the axiom of separation. This will also work if there are a countably infinite number of boxes. The problem arises when there is an uncountable number of sets because we can't order them in a way to pick from each one. The axiom of choice states that there is some way to choose an element from each set, even if there is an uncountable number of sets. Even if we don't know what is in the new, uncountable, set, the axiom of choice states that it exists. There are six official forms of the Axiom of Choice:

**6.1.** For any relation $R$, there is a function $F \subseteq R$ with $dom(F) = dom(R)$

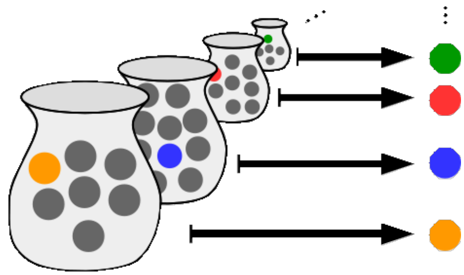

Figure 1: Axiom of Choice

**6.2.** If $H$ is a function, $I = dom(H)$ and for all $i \in I(H(i) \neq \emptyset)$, then $X_{i \in I} H(i) \neq \emptyset$

**6.3.** For every set $A$, there is a function $F : P(A)\{\emptyset\} \to A$ with $F(B) \in B$ for all $B \subseteq A$, $B \neq \emptyset$.

**6.4.** For every set $A$ of non-empty disjoint sets, there is a set $C$ such that for all $X \in A$, $|C \cap X| = 1$.

**6.5. Cardinal Comparability:** For any sets $C$, $D$, either $C \leftrightarrow D$ or $D \leftrightarrow C$

**6.6. Zorn's lemma:** If every totally ordered subset of a partially ordered set $S$ has an upper bound, then $S$ contains a maximal element.

# 7 Proof that Zorn's Lemma is equivalent to the Axiom of Choice

Let $A$ be a partially ordered set.

We say that $A$ is inductively ordered if every totally ordered subset $T$ of $A$ has an upper bound, i.e., an element $a \in A$ such that for all $x \in T$ , $x \leq a$ . We say that $A$ is *strictly inductively ordered* if every totally ordered subset $T$ of $A$ has a least upper bound, i.e., an upper bound $a$ so that if $b$ is an upper bound of $T$, then $a \leq b$.

An element $m \in A$ is maximal if the relation $a \geq m$ implies $a = m$. (A set may have several maximal elements.)

We say a function $f : A \to A$ is increasing if $x \leq f(x)$ for all $x \in A$

**Every inductively ordered set $A$ has a maximal element.**

*Proof (using the axiom of choice):*

Let $A$ be a strictly inductively ordered set, and let $f : A \to A$ be an increasing function. Pick some $a \in A$. Let $A'$ be the set of elements $x \in A$ such that $a \leq A$. Then, $A'$ is strictly inductively ordered, for if $T$ is a totally ordered subset of $A'$, then it has a least upper bound in $A$, which is greater than $a$, so this least upper bound is an element of $A'$. We say that a subset $B \subseteq A'$ is *admissable* if it satisfies these conditions:

- $a \in B$
- $f(B) \subseteq B$
- For every totally ordered subset $T \subseteq B$, the least upper bound of $T$ in $A'$ is an element of $B$.

Let $M$ be the intersection of all admissable subsets of $A'$. We note that $M$ is not empty, as $A'$ is an admissable subset of itself, and all admissable sets contain $a$. Then $M$ is the least admissable set, under order by inclusion.

We say that the element $c \in M$ is an extreme point if $x \in M$, $x < c$ together imply $f(x) \leq c$. For an extreme point $c$ denote by $M_c$ the set of $x \in M$ such that $x \leq c$ or $f(c) \leq x$.

**Lemma 7.1: For each extreme point $c$, $M_c = M$.** *Proof:*

It suffices to show that $M_c$ is an admissable set. Evidently, $a \leq c$, so $a \in M_c$ Now, let $x$ be an element of $M_c$. If $x = c$, then evidently, $f(c) \leq f(x)$, so $f(x) \in M_x$. If $x < c$ , then since $c$ is an extreme point, $f(x) \leq c$, so $f(x) \in M_c$. On the other hand, if $f(c) \leq x$, then $f(c) \leq x \leq f(x)$ , so $f(x) \in M_c$. Therefore $f(M_c) \subseteq M_c$.

Let $T$ be a totally ordered subset of $M_c$ . Then $T$ has a least upper bound $s \in A'$. Since $M$ is admissable, $s \in M$. Now, if $s \leq c$, then $s \in M_c$. On the other hand, if $s \geq c$, then either $f(c) \leq s$, or every element of $T$ is less than or equal to $c$, so $s \leq c$. Hence the least upper bound of every totally ordered subset $T$ of $M_c$ is an element of $M_c$, so $M_c$ is admissable. Therefore $M \subseteq M_c$; since we know $M_c \subseteq M$, it follows that $M = M_c$.

**Lemma 7.2: Every element of $M$ is an extreme point.** *Proof:*

Let $E$ be the set of the extreme points of $M$. As before, it suffices to show that $E$ is an admissable set. Evidently, $a$ is an extreme point of $M$, as no element of $M$ is less than $a$, so every element less than $a$ is also less than or equal to $f(a)$. Now, suppose $c$ is an extreme point of $M$. Then for any $x \in M$, if $x < f(c)$, then by Lemma 1, $x \leq c$. If $x = c$, then $f(x) = f(c)$, so $f(x) \leq f(c)$; if $x < c$, then since $c$ is an extreme point, $f(x) \leq c \leq f(c)$. Therefore f(c) is an extreme point, so $f(E) \subseteq E$.

Now, let $T$ be a totally ordered set of extreme points. Consider the least upper bound $s$ of $T$ in $M$. If $x$ is an element of $M$ strictly less than $s$, then $x$

must be strictly less than some element $c \in T$. But $c$ is an extreme point, so $f(x) \leq c \leq s$. Therefore $s$ is an extreme point, i.e., an element of $E$. It follows that $E$ is an admissable set, so as before, $E = M$.

**Theorem 7.3: For any strictly inductively ordered set $A$ and any increasing function $f : A \to A$ , there exists an element $x_0$ of such that $x_0 = f(x_0)$.** *Proof:*

Choose an arbitrary $a \in A$, and define $A'$ as before. Let $M$ be the least admissable subset of $A'$, as before. By Lemmas 2 and 1, for all elements $a, b \in M$, either $a \leq b$, or $b \leq f(b) \leq a$. Therefore $M$ is totally ordered under the ordering induced by $A$. Then $M$ has a least upper bound $x_0$ which is an element of $M$. We note that $f(x_0) \in M$, so $f(x_0) \leq x_0$, and since $f$ is increasing, $x_0 \leq f(x_0)$. Hence $x_0 = f(x_0)$, as desired.

**Corollary 7.4: Let $A$ be a strictly inductively ordered set. Then $A$ has a maximal element.**

*Proof:*

Suppose the contrary. Then by the Axiom of Choice, for each $x \in A$, we may define $f(x)$ to be an element strictly greater than $x$. Then $f$ is an increasing function, but for no $x \in A$ does $x = f(x)$, which contradicts theorem 7.3.

**Corollary 7.5 (Zorn's Lemma): Let $A$ be an inductively ordered set. Then $A$ has a maximal element.**

*Proof:*

Let $T$ be the family of totally ordered subsets of $A$.

We claim that under the order relation $\subseteq$, $T$ is a strictly inductively ordered set. If $\{X_i\}_{i \in I}$ is a totally ordered subset of $T$, then

$$Z = \bigcup_{i \in I} X_i$$

is the least upper bound of the $X_i$, and if $a, b \in Z$, then for some $i, j \in I$, $a \in X_i$ and $b \in X_j$; one of $X_i$ and $X_j$ is a subset of the other, by assumption, so $a$ and $b$ are comparable. It follows that $Z$ is totally ordered, i.e., $Z \in T$.

Now, by Corollary 7.4, there exists a maximal element $P$ of $T$. This set $P$ is totally ordered, so it has an upper bound $x_0$ in $A$. Then $P \cup \{x_0\}$ is a totally ordered set, so by the maximality of $P$, $x_0 \in P$. Now, if $y \geq x_0$, then $P \cup \{y\}$ is a totally ordered set, so $y \in P$ and $y \leq x_0$, so $y = x_0$. Therefore $x_0$ is a maximal element, as desired.

# 8 Ordinals

An ordinal number, $\alpha$, is defined as the set of all smaller ordinals, $\alpha = \{\beta | \beta < \alpha\}$, where $<$ is a strict-well ordering relation. Ordinals extend the natural numbers to include the position and order beyond finite counting.

**Definition 8.1:** An ordinal is a transitive set well-ordered under the relation $x < y$ if $x \in y$.

**Definition 8.2:** Let $\alpha$ and $\beta$ be ordinals. We write $\alpha < \beta$ if $\alpha \in \beta$.

**Definition 8.3:** Let $\alpha$ be an ordinal, and define the successor of $\alpha$, denoted by $\alpha + 1$, to be $\{\alpha\} \cup \alpha$.

**Proposition 8.4:** $\alpha + 1$ is an ordinal.

*Proof* Pick $x, y, z \in \alpha + 1$. If $x \in \alpha$, then $x \in x$ as $\alpha$ is an ordinal, while if $x = \alpha$ then $x \in x$ as $\alpha$ is strictly ordered. If $x \in y$ and $y \in z$, if $z = \alpha$ then $x < z$ as $\in$ is transitive on $\alpha$. If $z = \alpha$, then $x \in \alpha$. If $x, y \in \alpha$ then $x < y$, $y < x$, or $x = y$. If $x = \alpha$ but $y = \alpha$, then $x \in \alpha$ so $x < y$. If $y = \alpha \in x$, then as $\alpha$ is transitive, $\alpha \in \alpha$. But $\in$ is a strict ordering on $\alpha$, so $y \in x$. Finally, if $x = y = \alpha$ then $x \in y$ by the same reasoning. Therefore $\in$ is a strict ordering. $\alpha + 1$ is transitive, for if $y \in \alpha + 1$ then either $y = \alpha$ and $\alpha \subset \alpha + 1$, or $y \in \alpha$. But $\alpha$ is transitive, so $y \subset \alpha \subset \alpha + 1$. It is well-ordered: note that the only new element, $\alpha$, is greater than any $x \in \alpha$. Thus a non-empty subset of $\alpha + 1$ either contains only $\alpha$, in which case it has a least element, or the intersection with $\alpha$ is non-empty and hence has a least element as $\alpha$ is well-ordered.

**Definition 8.5:** A limit ordinal is an ordinal $\alpha$ such that there does not exist an ordinal $\beta$ with $\beta + 1 = \alpha$.

**Definition 8.6:** Let $\alpha$ be an ordinal, and $\beta \in \alpha$. Then $\beta$ is an ordinal.

*Proof* Since $\alpha$ is transitive, $\beta \subset \alpha$. Therefore, $\beta$ is well-ordered by $\in$ since $\alpha$ is. Now suppose $\gamma \in \beta$ and $\delta \in \gamma$. Since $\in$ is a strict total order on $\alpha$, $\delta \in \beta$. Therefore, $\beta$ is transitive.

**Definition 8.7:** Let $A$ be an initial segment of an ordinal $\alpha$ (this means it is a subset with the property that for $x \in A$ and $y \in \alpha$, $y < x$ implies $y \in A$). Then $A$ is an ordinal, and either $A \in \alpha$ or $A = \alpha$.

*Proof* First, we show that $A$ is an ordinal. If $x \in y$ and $y \in z$ for $x, y, z \in A$, then $x \in z$ as $A \subset \alpha$ is ordered. $A$ is well-ordered as any subset is also a subset of $\alpha$ which is well-ordered. $A$ is transitive, for if $y \in A$, as $A$ is initial, any $x$ with $x \in y$ also satisfies $x \in A$, thus $y \subset A$. Now suppose $A = \alpha$. For $\beta \in \alpha$, either $\beta \in A$ or $\beta > \gamma$ for every $\gamma \in A$. Thus there exists a $\beta$ such that $A \subset \beta$ as $\beta$ contains all smaller ordinals. Pick the least such $\beta$, and suppose $A = \beta$. Then there is a $\delta \in \beta$ with $A \subset \delta$. But $\alpha$ is transitive, so $\delta \in \alpha$. Thus $\beta$ is not minimal. Hence $A = \beta$, so $A \in \alpha$.

**Definition 8.8 (Trichotomy):** If $\alpha$ and $\beta$ are ordinals, either $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$.

*Proof* Given distinct ordinals $\alpha$ and $\beta$, let $A = \alpha \cap \beta$. $A$ is an initial segment of $\alpha$ since if $x \in \alpha$ and if $x < y \in \alpha$ then $x \in \alpha$. Likewise for $\beta$. By the lemma, either $A \in \alpha$ or $A = \alpha$. Likewise, either $A \in \beta$ or $A = \beta$. If $A \in \alpha$ and $A \in \beta$, then $A \in A$. However, this contradicts the fact that $\in$ is a strict order. Therefore, $\alpha < \beta$, $\beta < \alpha$, or $\alpha = \beta$.

**Definition 8.9:** Let $\omega$ be the smallest set of ordinals closed under successor such that $\emptyset \in \omega$.

**Definition 8.10:** $\omega$ is a limit ordinal.

*Proof* As $\omega$ is a set of ordinals, it is strictly well-ordered by $\in$. We need to show that it is transitive. If not, let $A = \{\alpha \in \omega : \alpha \subset \omega\}$. Let its least element be $\alpha^*$. Then $\alpha^* = \emptyset$. Suppose $\alpha^* = \beta + 1$ for some $\beta \in \omega$. Then $\beta \subset \omega$, and $\alpha^* = \beta \cup \{\beta\} \subset \omega$, which is a contradiction. Therefore, $\alpha^*$ is not a successor of anything in $\omega$. But $\omega$ is the smallest set containing $\emptyset$ and closed under successor. Suppose $\omega$ is a successor, so $\alpha + 1 = \omega$. Then $\alpha < \omega$, so $\alpha \in \omega$. Since $\omega$ is closed under taking successor, $\omega \in \omega$. But $\in$ is a strict ordering on any ordinal. Thus, $\omega$ is a limit ordinal.

# 9 Transfinite Induction

Induction is a technique used to prove statements about linearly ordered sets. It usually consists of two steps:

**Base Case:** Prove that the statement holds true for the simplest case, typically $n = 1$ or $n = 0$.

**Inductive Step:** Assume that the statement is true for an arbitrary but fixed $n = k$ ($k$ is an integer). Use this assumption to prove that the statement is also true for $n = k + 1$.

i.e. The sum of the first $n$ natural numbers is $\frac{n(n+1)}{2}$.

*Proof with induction:*

**Base Case:** For $n = 1$, $\frac{1 \cdot (1+1)}{2}$ , which is true.

**Inductive Step:** Assume true for $n = k$ $(1 + 2 + ...k = \frac{k(k+1)}{2})$. Show true for $n = k + 1$: $1 + 2 + ... + k + (k + 1) = \frac{(k+1)(k+2)}{2}$ .

Therefore, it holds for $n = k + 1$ if it holds for $n = k$, completing the inductive step.

**Transfinite Induction:**

Transfinite induction is an extension of regular that allows us to prove statements for infinitely many cases. Usually, it involves well-ordered sets like ordinal numbers.

*Proof with transfinite induction:*

**Base Case:** The base case of transfinite induction is the same as regular induction.

**Successor Case:** Assuming the statement is true for all previous elements up to $\alpha$, prove it for the next element, $\alpha + 1$.

**Limit Case (Transfinite Step)**: If $\alpha$ is a limit ordinal (a step beyond which there are no immediate successors), prove that the statement holds for all previous ordinals less than $\alpha$.

i.e. Consider proving a property $P(\alpha)$ for all ordinal numbers $\alpha$:

**Base Case:** Prove $P(0)$.

**Successor Case:** Assume $P(\alpha)$ holds, prove $P(\alpha + 1)$.

**Limit Case:** If $\alpha$ is a limit ordinal, assume $P(\beta)$ holds for all $\beta < \alpha$, prove $P(\alpha)$.

# 10 Transfinite Recursion

Recursion is when a function is defined in terms of itself, either directly, or indirectly. It is a method used to solve problems by breaking them down into smaller, similar problems and solving those smaller problems recursively until a base case is reached.

**Base Case:** This is the simplest form of the problem that can be directly solved without further recursion.

**Recursive Case:** This defines how the function behaves for larger instances of the problem, typically by reducing the problem size and applying the same function to the reduced problem.

i.e. Factorial numbers use recursion:

**Base Case:** $0! = 1$

**Recursive Case:** $n! = n(n - 1)!$

**Transfinite Recursion**:

Transfinite recursion extends recursion to more than just the natural numbers. It is used on ordinals.

An example of transfinite recursion is ordinal arithmetic.

# 11 Ordinal Arithmetic

We shall now define addition, multiplication and exponentiation of ordinal numbers, using Transfinite Recursion.

**Definition 11.1 (Addition):** For all ordinal numbers $\alpha$
**1.** $\alpha + 0 = \alpha$
**2.** $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all $\beta$,
**3.** $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$.
**Definition 11.2 (Multiplication):** For all ordinal numbers $\alpha$
**1.** $\alpha \cdot 0 = 0$,
**2.** $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all $\beta$,
**3.** $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$.
**Definition 11.3 (Exponentiation):**
**1.** $\alpha^0 = 1$,
**2.** $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all $\beta$,
**3.** $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$.

As defined, the operations $\alpha + \beta$, $\alpha \cdot \beta$ and $\alpha^\beta$ are normal functions in the second variable $\beta$. Their properties can be proved by transfinite induction.

For all ordinals $\alpha$, $\beta$, and $\gamma$ $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

*Proof* By induction on $\gamma$.

Neither + nor $\cdot$ are commutative:

$$1 + \omega = \omega = \omega + 1, \quad 2 \cdot \omega = \omega = \omega \cdot 2 = \omega + \omega.$$

# 12 Cantor's Normal Form Theorem

**Lemma 12.1** **i** If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$.
**ii** If $\alpha < \beta$ then there exists a unique $\delta$ such that $\alpha + \delta = \beta$.
**iii** If $\beta < \gamma$ and $\alpha > 0$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.
**iv** If $\alpha > 0$ and $\gamma$ is arbitrary, then there exist a unique $\beta$ and a unique $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.
**v** If $\beta < \gamma$ and $\alpha > 1$, then $\alpha\beta < \alpha\gamma$.

*Proof.* (i), (iii) and (v) are proved by induction on $\gamma$.
(ii) Let $\delta$ be the order-type of the set $\{\xi : \alpha \leq \xi < \beta\}$; $\delta$ is unique by (i).
(iv) Let $\beta$ be the greatest ordinal such that $\alpha \cdot \beta \leq \gamma$.
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**Theorem 12.2 (Cantor's Normal Form Theorem)** Every ordinal $\alpha > 0$ can be represented uniquely in the form

$$\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,$$

where $n \geq 1$, $\alpha \geq \beta_1 > \ldots > \beta_n$, and $k_1, \ldots, k_n$ are nonzero natural numbers.

*Proof* By induction on $\alpha$. For $\alpha = 1$ we have $1 = \omega^0 \cdot 1$; for arbitrary $\alpha > 0$ let $\beta$ be the greatest ordinal such that $\omega^\beta \leq \alpha$. By Lemma 2.25(iv) there exists a unique $\delta$ and a unique $\rho < \omega^\beta$ such that $\alpha = \omega^\beta \cdot \delta + \rho$; this $\delta$ must necessarily be finite.

## References


[End77] Herbert B Enderton. *Elements of set theory*. Academic press, 1977.

[Jec03] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[Lan12] Serge Lang. *Algebra*, volume 211. Springer Science & Business Media, 2012.

[Sha11] Daniel Shanks. Constructing the integers: N, ordinal numbers, and transfinite arithmetic. 2011.

[Wei08] William AR Weiss. An introduction to set theory, 2008.


[End77] [Jec03] [Sha11] [Wei08] [Lan12]