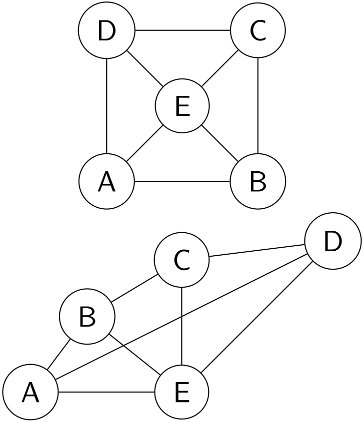
On Asymmetric Graph Coloring Games in Undirected and Oriented Forests

Kshemaahna Nagi

July 2024

Table of Contents

Some Graph Theory Background



Are these 2 graphs the same?

Some Graph Theory Background

Definition

A graph is an ordered pair G (V, E) consisting of a nonempty set V (called the vertices) and a set E (called the edges) of two-element subsets of V.

Some Graph Theory Background

Definition

A graph is an ordered pair G (V, E) consisting of a nonempty set V (called the vertices) and a set E (called the edges) of two-element subsets of V.

Thus, the 2 graphs shown before can both be represented as: $G = \{V, E\}$ where $V = \{A, B, C, D, E\}$ and
$E = \{(A, B), (A, E), (A, D), (B, C), (B, E), (C, D), (C, E), (D, E)\}$.

Some Graph Theory Background

Definition

A graph is an ordered pair G (V, E) consisting of a nonempty set V (called the vertices) and a set E (called the edges) of two-element subsets of V.

Thus, the 2 graphs shown before can both be represented as: $G = \{V, E\}$ where $V = \{A, B, C, D, E\}$ and
$E = \{(A, B), (A, E), (A, D), (B, C), (B, E), (C, D), (C, E), (D, E)\}$.

An Edge (u,v) can also be written as uv.

Some Graph Theory Background

Definition

A graph is an ordered pair G (V, E) consisting of a nonempty set V (called the vertices) and a set E (called the edges) of two-element subsets of V.

Thus, the 2 graphs shown before can both be represented as: $G = \{V, E\}$ where $V = \{A, B, C, D, E\}$ and $E = \{(A, B), (A, E), (A, D), (B, C), (B, E), (C, D), (C, E), (D, E)\}$.

An Edge (u,v) can also be written as uv.
uv or vu: are they the same?

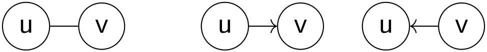
Some Graph Theory Background

Definition

A graph is an ordered pair G (V, E) consisting of a nonempty set V (called the vertices) and a set E (called the edges) of two-element subsets of V.

Thus, the 2 graphs shown before can both be represented as: $G = \{V, E\}$ where $V = \{A, B, C, D, E\}$ and
$E = \{(A, B), (A, E), (A, D), (B, C), (B, E), (C, D), (C, E), (D, E)\}$.

An Edge (u,v) can also be written as uv.
uv or vu: are they the same?



In an undirected graph: yes. In a directed graph: no.

Some Graph Theory Background

Definition

The number of vertices in a graph G is called the order of G, denoted by n or $|G|$.

Definition

The degree of a vertex v in a graph G is the number of vertices in G that are adjacent to v.

The largest degree of any vertex v $\in$ G is called the maximum degree of G and is denoted by Δ(G). The minimum degree of any vertex v $\in$ G is denoted by δ(G).

$$0 \leq \delta(G) \leq deg(v) \leq \Delta(v) \leq n-1$$

Some Graph Theory Background

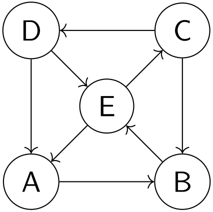
Degrees of a directed graph/digraph:

- In-degree is the number of edges directed *towards* a vertex.
- Out-degree is the number of edges directed *away from* a vertex.
- The degree of a vertex equals the sum of its in-degree and out-degree.

Some Graph Theory Background

Degrees of a directed graph/digraph:

- In-degree is the number of edges directed *towards* a vertex.
- Out-degree is the number of edges directed *away from* a vertex.
- The degree of a vertex equals the sum of its in-degree and out-degree.



Some Graph Theory Background

Now, we discuss 2 special classes of graphs.

Definition

A tree is a connected acyclic graph.

Thus, there is a unique path joining any two vertices of a tree.

Theorem

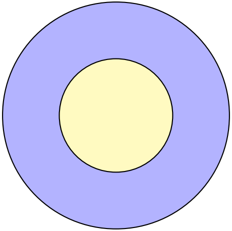
All trees have leaves.

Some Graph Theory Background

Definition

Forests are defined as acyclic graphs.

Thus, forests consist of only trees. The trees making up a forest may be disconnected. The class forests is represented by $\mathcal{F}$ while an individual graph belonging to the forests class is represented by F.



Some Graph Theory Background

A graph is connected if there is atleast one path between any 2 vertices.

Definition

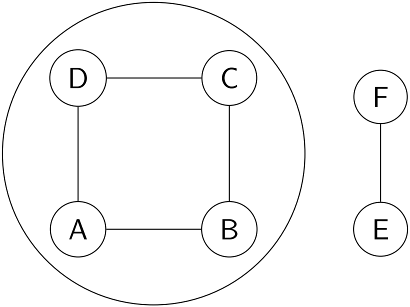
A maximal connected sub graph is a connected sub graph of a graph to which no vertex can be added and it still be connected.

Some Graph Theory Background

A graph is connected if there is atleast one path between any 2 vertices.

Definition

A maximal connected sub graph is a connected sub graph of a graph to which no vertex can be added and it still be connected.



In the above graph, one of the maximal connected sub graphs is circled.

Graph Coloring Games

1. SIMPLE VERTEX COLORING GAME OR t-COLORING GAME
   - This game is played on a finite graph G = (V,E).

Graph Coloring Games

1. SIMPLE VERTEX COLORING GAME OR t-COLORING GAME
   - This game is played on a finite graph G = (V,E).
   - Players can use colors from a palette or a set C that contains t colors. What this means is that, in set C, each color is assigned an integer label from $\{1, 2, \ldots, t\}$.

Graph Coloring Games

1. SIMPLE VERTEX COLORING GAME OR t-COLORING GAME
    - This game is played on a finite graph G = (V,E).
    - Players can use colors from a palette or a set C that contains t colors. What this means is that, in set C, each color is assigned an integer label from $\{1, 2, \ldots, t\}$.
    - In this game, a color is said to be legal for a vertex v $\in$ V if it belongs to the palette and there is no neighbour of v which has been assigned the same color.

Graph Coloring Games

1. SIMPLE VERTEX COLORING GAME OR t-COLORING GAME

- This game is played on a finite graph G = (V,E).
- Players can use colors from a palette or a set C that contains t colors. What this means is that, in set C, each color is assigned an integer label from $\{1, 2, \ldots, t\}$.
- In this game, a color is said to be legal for a vertex v $\in$ V if it belongs to the palette and there is no neighbour of v which has been assigned the same color.
- The game is played by two players whom we call Alice and Bob.

Graph Coloring Games

1. SIMPLE VERTEX COLORING GAME OR t-COLORING GAME
   - This game is played on a finite graph G = (V,E).
   - Players can use colors from a palette or a set C that contains t colors. What this means is that, in set C, each color is assigned an integer label from $\{1, 2, \ldots, t\}$.
   - In this game, a color is said to be legal for a vertex v $\in$ V if it belongs to the palette and there is no neighbour of v which has been assigned the same color.
   - The game is played by two players whom we call Alice and Bob.
   - The two players alternate turns coloring one vertex each turn with a legal color.

Graph Coloring Games

1. SIMPLE VERTEX COLORING GAME OR t-COLORING GAME
   - This game is played on a finite graph G = (V,E).
   - Players can use colors from a palette or a set C that contains t colors. What this means is that, in set C, each color is assigned an integer label from $\{1, 2, \ldots, t\}$.
   - In this game, a color is said to be legal for a vertex v $\in$ V if it belongs to the palette and there is no neighbour of v which has been assigned the same color.
   - The game is played by two players whom we call Alice and Bob.
   - The two players alternate turns coloring one vertex each turn with a legal color.
   - Alice begins the game.

Graph Coloring Games

1. SIMPLE VERTEX COLORING GAME OR t-COLORING GAME
   - This game is played on a finite graph G = (V,E).
   - Players can use colors from a palette or a set C that contains t colors. What this means is that, in set C, each color is assigned an integer label from $\{1, 2, \ldots, t\}$.
   - In this game, a color is said to be legal for a vertex v $\in$ V if it belongs to the palette and there is no neighbour of v which has been assigned the same color.
   - The game is played by two players whom we call Alice and Bob.
   - The two players alternate turns coloring one vertex each turn with a legal color.
   - Alice begins the game.
   - Alice wins the game if every vertex of G has been assigned a legal color. Bob wins if there is at least one vertex which cannot be assigned a legal color.

Graph Coloring Games

In short: Alice and Bob alternate turns coloring vertices ensuring that no two adjacent vertices have the same color.

Graph Coloring Games

In short: Alice and Bob alternate turns coloring vertices ensuring that no two adjacent vertices have the same color.

Definition

The game chromatic number of the playing graph G, denoted by $\chi_g(G)$, is defined as the least t for which Alice has a winning strategy when the simple vertex coloring game is played on G.

Graph Coloring Games

2. ASYMMETRIC VERTEX COLORING GAME

The (a,b)-coloring game or the asymmetric coloring game is an altered version of the t-coloring game wherein Alice must color a vertices and Bob must color b vertices on a single turn. The rest of the rules are the same.

2. ASYMMETRIC VERTEX COLORING GAME

The (a,b)-coloring game or the asymmetric coloring game is an altered version of the t-coloring game wherein Alice must color a vertices and Bob must color b vertices on a single turn. The rest of the rules are the same.

Definition

The (a,b)-game chromatic number of a graph G is the minimum number of colors required in the palette such that Alice has a winning strategy when the (a,b)-coloring game is played with the palette on G. It is denoted by $\chi_g(G; a, b)$.

2. ASYMMETRIC VERTEX COLORING GAME

The (a,b)-coloring game or the asymmetric coloring game is an altered version of the t-coloring game wherein Alice must color a vertices and Bob must color b vertices on a single turn. The rest of the rules are the same.

Definition

The (a,b)-game chromatic number of a graph G is the minimum number of colors required in the palette such that Alice has a winning strategy when the (a,b)-coloring game is played with the palette on G. It is denoted by $\chi_g(G; a, b)$.

Now, we define the (a,b)-coloring game chromatic number for a class of graphs, say $\mathcal{C}$:

Definition

$\chi_g(\mathcal{C}; a, b) := \max_{G \in \mathcal{C}} \ \chi_g(G; a, b)$

3. MODIFIED COLORING GAME (MCG) OR t-MCG

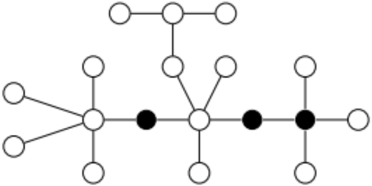
This game is played on partially colored forests (forests with at least one vertex colored or labelled with an integer). The rules are the same as the t-coloring game with two exceptions:

- Bob begins or plays the first turn
- Bob can choose to pass a turn

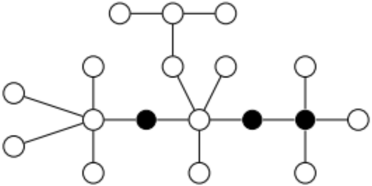
Definition

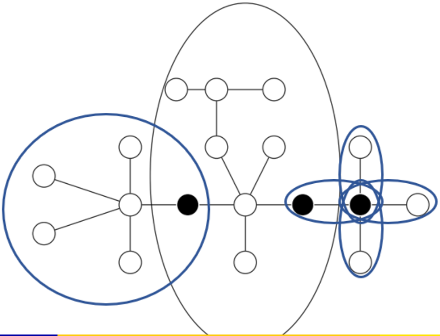
Let F be a partially colored forest. R is a trunk of F if R is a maximal connected sub graph of F such that every colored vertex in R is a leaf of R. $\mathcal{R}$(F) is the number of such trunks R on F.

Graph Coloring Games



Graph Coloring Games





Simple Vertex Coloring Game on Forests

Theorem

$\chi_g(F) \leq 4$

Simple Vertex Coloring Game on Forests

Theorem

$\chi_g(F) \leq 4$

A lemma given without proof:

Lemma

Let F be a partially colored forest. If Alice can win the t-MCG on every trunk in $\mathcal{R}(F)$ she can win the t-coloring game on $\mathcal{R}(F)$.

Simple Vertex Coloring Game on Forests

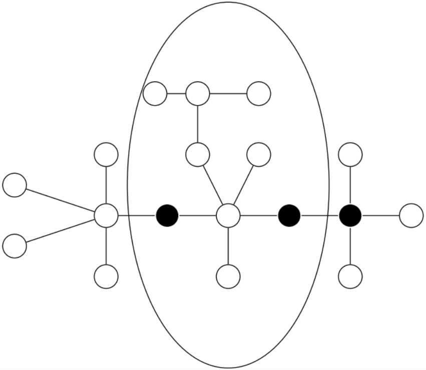
PROOF IDEA: Alice can play in a way such that every uncolored vertex has at most 3 colored neighbors, thereby guaranteeing every uncolored vertex has an available legal color if 4 colors are used.

Let us take a trunk from the partially colored forest shown below.

Simple Vertex Coloring Game on Forests

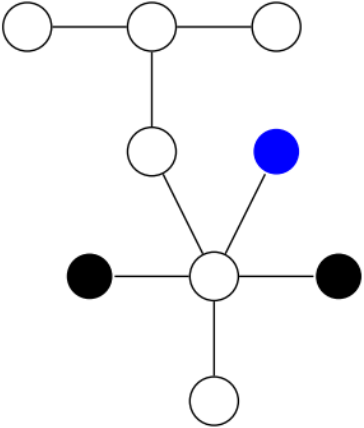
PROOF IDEA: Alice can play in a way such that every uncolored vertex has at most 3 colored neighbors, thereby guaranteeing every uncolored vertex has an available legal color if 4 colors are used.

Let us take a trunk from the partially colored forest shown below.

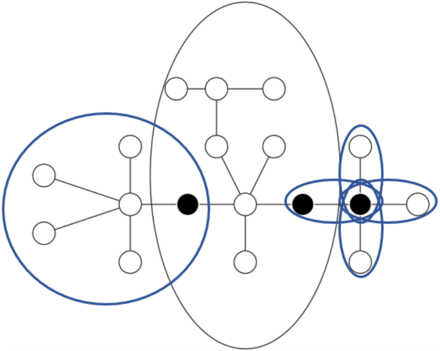


Simple Vertex Coloring Game on Forests

After Bob's first turn:



If the game is played on any other trunk:



Asymmetric Vertex Coloring Game on Undirected Forests

Theorem

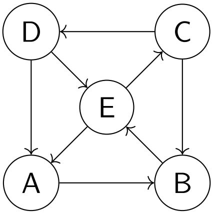
Let a and b be positive integers. Then,
(a) For $a < b$, $\chi_g(\mathcal{F}; a, b) = col_g(\mathcal{F}; a, b) = \infty$.
(b) For $b \leq a$, $b+2 \leq \chi_g(\mathcal{F}; a, b) \leq col_g(\mathcal{F}; a, b) \leq b+3$.
(c) For $b \leq a < max\{2b, 3\}$, $b + 3 \leq \chi_g(\mathcal{F}; a, b)$.
(d) For $4 \leq 2b \leq a < 3b$, $\chi_g(\mathcal{F}; a, b) \leq b+2 < b+3 \leq col_g(\mathcal{F}; a, b)$

Asymmetric Vertex Coloring Game on Oriented Forests

The Game is modified.

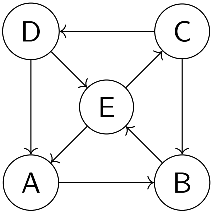
Asymmetric Vertex Coloring Game on Oriented Forests

The Game is modified.



Asymmetric Vertex Coloring Game on Oriented Forests

The Game is modified.



Theorem

Let a and b be positive integers. Then,
(a) for $b \leq a : \chi_g(\overrightarrow{\mathcal{F}}; a, b) = col_g(\overrightarrow{\mathcal{F}}; a, b) = b + 2$
(b) for $a < b : \chi_g(\overrightarrow{\mathcal{F}}; a, b) = col_g(\overrightarrow{\mathcal{F}}; a, b) = \infty$

Thank You

Thank you so much for your time.