ON ASYMMETRIC GRAPH COLORING GAMES IN UNDIRECTED AND ORIENTED FORESTS

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1. Abstract

Here we explore some central results of graph coloring games on both undirected and oriented forests. First, we show that the game chromatic number of undirected forests is at most 4. Then we discuss the upper and lower bounds of the (a,b)-game chromatic number of undirected forests for different cases. We also take the special case of the (1,1)-game chromatic number or simply, the game chromatic number. Finally, we discuss the upper and lower bounds for the (a,b)-game chromatic number on oriented forests.

2. Introduction

Graph coloring games are usually two-player games where the players named Alice and Bob alternate turns coloring the vertices of a graph such that no two adjacent vertices are assigned the same color. The game chromatic number was first explored by Bodlaender [Bod91]. When both players color a single vertex on each turn, it is known as the simple vertex coloring game. However, when Alice and Bob color a vertices and b vertices respectively, such that a $\geq$ 1 and b $\geq$ 1, it is known as the asymmetric coloring game. The (a,b)-game chromatic number is the least number of colors required for Alice to have a winning strategy regardless of how Bob moves. The (1,1)-game chromatic number is also known simply as the game chromatic number for the simple vertex coloring game. Forests are a class of acyclic graphs. In this paper, we first show that the game chromatic number of forests can be at most 4. Then we establish upper and lower bounds for the (a,b)-game chromatic number of undirected forests and finally oriented forests. The theorems for the (a,b)-game chromatic number for undirected forests were established by Kierstead [Kie05] and the theorems for the (a,b)-game chromatic number for oriented forests were established by Andres [And09].

3. Background

Relevant Graph Theory Background

Definition 3.1. A graph is defined as an ordered pair G = (V,E) where V is a non-empty set called the vertices and E is a set of 2 element subsets of V called the edges.

In the scope of this paper, the term graphs is taken to refer to simple graphs, which places the following additional constraints on the definition above:

(1) No vertex is connected to itself. In other words, for all v $\in$ V, no element in E can be of the form (v,v).
(2) Multiple edges between any given two vertices are not permitted.

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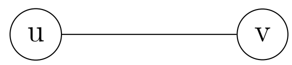
Let there be two vertices u and v in the vertex set of a graph G. Say (u,v) belongs to the edge set of G. The question arises whether (u,v) is an ordered pair, in other words, whether (u,v) and (v,u) are the same.

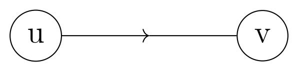
The answer depends on whether G is an undirected graph or a directed graph.

An undirected graph is a graph whose edges are undirected. In other words, the edges can be traversed in any direction. The vertices are simply connected to each other. An example of an undirected edge is shown here.



Hence, for an undirected graph the edge (u,v) is not an ordered pair. (u,v) and (v,u) represent the same element of E or the edge set. Henceforth, we can use the unordered pair notation {u,v} for such edges.

A directed graph is a graph with directed edges. A directed edge is an edge which can only be traversed in one direction. An example is as shown below.



Hence, for a directed graph (u,v) is an ordered pair i.e. (u,v) and (v,u) represent different edges. (u,v) is directed from u towards v. The example edge shown above is (u,v). Such a directed edge is called an arc and a graph of directed edges is called a digraph or a directed graph.

Since the elements (u,v) and (v,u) are distinct from each other, they can both belong to the edge set of a simple digraph.

A directed graph is called an oriented graph if at most 1 arc between (u,v) and (v,u) exist in the edge set of the graph for all u,v in the vertex set of the graph.

Definition 3.2. Let G=(V,E). For any v $\in$ V, The neighborhood of a vertex v is the set of vertices adjacent to v or the set of vertices that share an edge with v. It is denoted as $N_G(v)$.

In other words, $N_G(v)$ is the set of vertices that occur as two element subsets alongside v in E. The vertex v itself is not a part of $N_G(v)$.

Definition 3.3. Let G be a graph (V,E) and $N_G\,[v]$ be the closed neighborhood of a vertex v $\in$ V. Then,

$$N_G\,[v] = N_G(v) \cap \{v\}$$

Definition 3.4. The order of a graph G is equal to the number of the vertices in the graph or the cardinal number of the vertex set of the graph. It is denoted by n or $|G|$.

Definition 3.5. Let G=(V,E). The degree of a vertex v $\in$ V is the number of vertices in N_G (v).

$$deg(v) = |N_G(v)|$$

In other words, the degree of a vertex is the number of vertices adjacent to it or the number of two element subsets of E the vertex occurs in.

The largest or maximum degree of any vertex v $\in$ V is denoted by Δ(G). The minimum degree of any vertex v $\in$ V is denoted by δ(G). From this we can infer the following inequality:

$$0 \leq \delta(v) \leq deg(v) \leq \Delta(v) \leq n - 1$$

Since we are considering simple graphs, any vertex may be connected to at most n-1 vertices, hence the upper bound.

If a vertex v belongs to the vertex set of a digraph G, then we define two more terms:

Definition 3.6. The in degree of a vertex is the number of edges directed towards the vertex in a digraph. It is denoted by d^{-}(v).

Definition 3.7. The out degree of a vertex is the number of edges directed away from the vertex in a digraph. It is denoted by d^{+}(v).

Hence, we see that

$$\deg(\mathrm{v}) = d^{-}(\mathrm{v}) + d^{+}(\mathrm{v}).$$

A graph is said to be connected if there is at least one path joining any two vertices u and v such that u,v belong to the vertex set of the graph.

Let G=(V,E) be a graph. A sub graph is a graph G'=(V',E') such that V' $\subseteq$ V and E' $\subseteq$ E.

A maximally connected sub graph is a subgraph which has the following properties:

- If u and v are present in V' then any edge connecting them present in E is also present in E'.
- Addition of even a single vertex would mean that the graph would be no longer connected.

Definition 3.8. A proper coloring of a graph is a coloring such that no two adjacent vertices have the same color. In other words, assigning colors to the graph avoiding monochromatic edges.

Lastly, we define some terms needed in the subsequent discussion of special classes of graphs.

Definition 3.9. A walk is a sequence of vertices and the edges between them in a graph.

A walk on a simple graph can be expressed simply as a sequence of vertices since each pair of consecutive vertices in such a sequence can share only one edge.

Definition 3.10. A trail is a walk without the repetition of edges.

Now, we discuss some special classes of graphs.

3.0.1. *Cycles.* A cycle is a graph consisting of vertices arranges in a non-empty trail with its initial and final vertices coinciding.

3.0.2. *Trees.* A tree is an acyclic, connected graph or a connected graph without cycles. All trees have leaves or pendant vertices (vertices with degree 1).

3.0.3. *Forests.* Forests are defined as acyclic graphs. Thus, forests consist of only trees. The trees making up a forest may be disconnected. The class forests is represented by F while an individual graph belonging to the forests class is represented by F.

Definition 3.11. R is a trunk of a forest F if R is a maximal connected subgraph of F such that every colored vertex in R is a leaf of R. $\mathcal{R}$(F) is the set of trunks on F.

3.0.4. *Stars.* The stars are trees on n vertices with one vertex having vertex degree n-1 and the other n-1 vertices having degree 1.

3.1. **Graph Coloring Games.** First, we discuss some facts that hold for the graph coloring games that will subsequently be discussed:

- At all turns during the games, both players now the current position and on working backwards past and future positions as well. This is known as complete information.
- The games are 2-player games and both players are said to employ optimal strategy while playing.

In such graph coloring games, a winning position is one from which a player can guarantee a winning strategy after it has been played.

Now, we introduce the four graph coloring games discussed in this paper, namely the t-coloring game, the Modified Coloring Game (MCG), the (a,b)-coloring game and the (a,b)-marking games. We also discuss the digraph versions of the (a,b)-coloring and (a,b)-marking games here. The MCG and (a,b)-marking games have been introduced in order to prove theorems relating to the t-coloring and (a,b)-coloring games respectively.

3.2. **The t-Coloring Game.** We begin our exploration with the simple vertex coloring game, also known as the t-coloring game. This game is played on a finite graph G = (V,E). Players can use colors from a palette or a set C that contains t colors. What this means is that, in set C, each color is assigned an integer label from $\{1, 2, \ldots, t\}$. In this game, a color is said to be legal for a vertex v ∈ V if it belongs to the palette and there is no neighbour of v which has been assigned the same color.

The game is played by two players whom we call Alice and Bob. The two players alternate turns coloring one vertex each turn with a legal color. Alice begins the game. Alice wins the game if every vertex of G has been assigned a legal color. Bob wins if there is at least one vertex which cannot be assigned a legal color. Now, we define the game chromatic number:

Definition 3.12. The game chromatic number of the playing graph G, denoted by $\chi_g(G)$, is defined as the least t for which Alice has a winning strategy when the simple vertex coloring game is played on G.

The following inequality helps us constrain the game chromatic number of G with upper and lower bounds, by establishing the relationship between the chromatic number, game chromatic number and the maximum degree possessed by any vertex belonging to the graph G as follows:

$$\chi(G) \leq \chi_g(G) \leq \Delta(\mathrm{G}) + 1$$

.

In this inequality, χ_g(G) has an intuitive upper and lower bound for the following reasons:

- **Lower Bound:**
  Recall definition of $\chi_g(G)$. the chromatic number of a graph is the least number of colors that are needed in the palette to obtain a proper coloring of the graph. Since the game play requires that Alice wins if a proper coloring of the graph is obtained at the end, the game chromatic number must be at least as large as the chromatic number of the playing graph.

- **Upper Bound:** Let v be the vertex in the playing graph G with maximum degree i.e. deg(v) = Δ(G). It may occur that during the game play, that every vertex neighboring v has been assigned a different color and v is uncolored. Hence, to obtain a proper coloring, Δ(G) + 1 colors would be required. Since, v has the maximum number of neighbors that any vertex on G can have, the maximum number of colors required to obtain a proper coloring on the graph during any game play would be Δ(G)+1.

However, this upper bound given by the inequality is sub-optimal for many classes of G, including forests. In Section 4, we explore a tighter upper bound for the game chromatic number of forests.

3.3. **The (a,b)-coloring game.** The (a,b)-coloring game or the asymmetric coloring game is a modified version of the t-coloring game wherein Alice must color a vertices and Bob must color b vertices on a single turn. On the last move of the game, if the number of uncolored vertices left is lesser than the vertices the player is required to color on a single turn, then the player going last is not required to complete their turn. Now we define the (a,b)-game chromatic number:

Definition 3.13. The (a,b)-game chromatic number of a graph G is the minimum number of colors required in the palette such that Alice has a winning strategy when the (a,b)-coloring game is played with the palette on G. It is denoted by $\chi_g(G; a, b)$.

Now, we define the (a,b)-coloring game chromatic number for a class of graphs, say $\mathcal{C}$:

Definition 3.14. $\chi_g(\mathcal{C}; a, b) := \max_{G\in\mathcal{C}} \ \chi_g(G; a, b)$

It is also interesting to note that such a game might be played between two players in order to obtain a coloring which has the minimum number of colors meeting a certain aesthetic standard. Here, aesthetic standard can be loosely thought of as a quantity proportional to the number of colors used.

Thus, Alice's goal is to use the minimum colors to obtain a proper coloring while Bob's goal is to effectively increase the number of colors used so the aesthetic standard is met.

In an ordinary coloring, 100 % of the effort is dedicated to obtaining a proper coloring with the minimum colors while in a simple vertex coloring game, 50% of the effort is dedicated to obtaining a proper coloring with minimum colors (Alice's efforts) and 50% of the efforts are dedicated to meeting the aesthetic standard (Bob's efforts).

In an (a,b)-coloring game, $\frac{100\times a}{a+b}$ % of the efforts are dedicated to obtaining a proper coloring with minimum colors while $\frac{100\times b}{a+b}$ % efforts are dedicated to meeting the aesthetic standard.

3.3.1. *Simple Vertex Coloring Game as a Special Case of the Asymmetric Coloring Game.* We can also note that the simple vertex coloring game or t-coloring game is a special case of the (a,b)-coloring game or the asymmetric game wherein both a and b are equal to 1. Thus, the (1,1)-game chromatic number is simply the game chromatic number and

$$\chi_g(\mathcal{C}; 1, 1) := \max_{G\in\mathcal{C}} \ \chi_g(G; 1, 1) \quad \text{or} \quad \chi_g(\mathcal{C}) := \max_{G\in\mathcal{C}} \ \chi_g(G)$$

for any class of graphs $\mathcal{C}$.

3.4. **The Marking Game.** The marking game is a two-player game between Alice and Bob where Alice begins and they subsequently alternate turns. On each turn, the player whose turn it is marks a vertex. Let's say the player marks a vertex v $\in$ V where V is the vertex set of the playing graph. The score assigned to v is equal to the number of neighbors of v which have been previously marked. After all the vertices have been marked, the score assigned to the game is the maximum score assigned to any vertex of the playing graph during the game play. Alice's goal is to minimise the score while Bob's goal is to maximise it. On their turn, a player can choose to mark any unmarked vertices without any restrictions.

A linear order L of the vertices is produced during the game play that correlates with the order in which the vertices of the playing graph G are marked. Alice is trying to minimise the number of neighbors of a vertex v that precede it in the linear order L.

Now, we define a few important terms, including a formal definition of the score at the end of the game.

Definition 3.15. Let a marking game be played on a graph G=(V,E) having a linear order L. For a vertex v $\in$ V, the number of vertices in $N_G(v)$ marked before v is marked is denoted by $N^+_{G_L}[v]$.

Definition 3.16. Let the score of a marking game on a playing graph G=(V,E) with linear order L be s. Then,

$$s = \max_{v \in V} |N^+_{G_L}[v]|$$

Definition 3.17. Let $\pi(G)$ be the set of all possible linear orders of vertices of a graph G=(V,E) obtained during the marking game. The coloring number col(G) is defined as:

$$\text{col(G)} = \min_{L \in \pi(G)} \left(\max_{v \in V} |N^+_{G_L}[v]| \right)$$

In other words, the game coloring number, col_g(G) is the minimum number k such that Alice can ensure that for a given vertex at most k-1 neighbors would be marked before it is marked.

3.5. **The Modified Coloring Game (MCG).** The Modified Coloring Game (MCG) also called the t-Modified Coloring Game (t-MCG) is played on partially colored forests (forests with at least one vertex colored or labelled with an integer). The rules are the same as the t-coloring game with two exceptions:

- Bob begins or plays the first turn
- Bob can choose to pass a turn

3.6. **Note:** The reason the playing graph G is assumed to be a simple graph can now be better understood since we have introduced the graph coloring games and the concept of a proper graph coloring.

Recall the definition 3.1 and its constraints for a simple graph. In the context of graph coloring games, the constraints are helpful as:

(1) **No vertex is connected to itself**
This is important as all the games discussed ensure a proper coloring of the playing graph G at every step. In other words, adjacent vertices cannot have the same color during the game play. Thus, a vertex cannot be adjacent to itself and meet this condition.

(2) **Multiple edges do not exist between the same pair of vertices**
A proper coloring of playing graph G is obtained on each turn. Thus, there are no monochromatic edges. Let there be multiple edges between 2 adjacent vertices belonging to G, say u and v. If even one edge between u and v is not monochromatic, all edges between u and v are not monochromatic. In other words, the presence of multiple edges in the playing graph does not affect the play of a vertex coloring game.

4. Simple Vertex Coloring Game on Undirected Forests

[DLL+15] In this section, we explore the simple vertex coloring game for a graph G belonging to forests.

Theorem 4.1. *The Game Chromatic Number of Forests is at most 4.*

Let $\mathcal{F}$ represents forests as a class of graphs. Then the theorem can be restated as follows:

$$\chi_g(\mathcal{F}; 1, 1) = 4 \quad or \quad \chi_g(\mathcal{F}) = 4$$

With this definition in mind, we will now prove some lemmas that will help us establish a proof of this theorem.

Lemma 4.2. *Let F be a partially colored forest. If Alice can win the t-MCG on $\mathcal{R}$(F) then she can win the t-coloring game on F.*

Proof. Recall that each trunk in $\mathcal{R}$(F) is a maximally connected sub graph such that every colored vertex is a leaf. Hence, each uncolored vertex occurs in only one trunk. In other words, each uncolored vertex in $\mathcal{R}$(F) has the same neighbors as in F. Hence, all Alice needs to do is use her winning strategy on the t-MCG on the t-coloring game assuming that the t-coloring game is a t-MCG where Bob chose to pass on his first turn and never passed his turn since. ■

Lemma 4.3. *Let F be a partially colored forest. If Alice can win the t-MCG on every trunk in $\mathcal{R}$(F) she can win the t-coloring game on $\mathcal{R}$(F).*

Proof. When Alice and Bob are playing the t-MCG on F, Alice pretends a separate game is being played on each trunk in $\mathcal{R}$(F). Thus, after Bob plays his turn, Alice continues with her winning strategy on the same trunk provided there is an uncolored vertex left on the trunk. If Bob passes or there is no uncolored vertex left, then Alice begins on the next trunk with an uncolored vertex and utilizes her winning strategy for that trunk. Alice wins the t-MCG on each trunk in $\mathcal{R}$(F) since a proper coloring is obtained on each trunk and by this she wins the t-MCG on $\mathcal{R}$(F). ■

Lemma 4.4. *Let F be a partially colored forest and suppose Alice and Bob are playing the 4-MCG on $\mathcal{R}$(F). If every trunk in $\mathcal{R}$(F) has at most two colored vertices, Alice can win the 4-MCG on $\mathcal{R}$(F).*

Proof. The 4-MCG implies that the game is being played with 4 colors in the palette. Thus, the proof idea is that if the MCG is played on a partially colored forest F, Alice can play in a way such that every uncolored vertex has at most 3 colored neighbors, thereby guaranteeing every uncolored vertex has an available legal color if 4 colors are used.

Let n be the number of uncolored vertices in $\mathcal{R}$(F).
When n=0,

Alice has already won the game as there are no uncolored vertices left or a proper coloring has already been obtained.

When n>0,
Case 1: Position 0: Bob colors a vertex on a trunk in $\mathcal{R}$(F) that has 2 colored vertices.

Every connected sub graph of a forest is a tree. By the definition of a trunk, a trunk is also a tree. Two leaves of such a trunk would be colored.

Bob plays his turn, by coloring either a leaf or another vertex.

Position 1: It is Alice's turn.

Arrangement of Playing Graph:

After Bob plays his turn, at most one trunk in $\mathcal{R}$(F) has 3 colored leaves. This would be the trunk which Bob colored his first turn on should he choose to color a leaf.

In such a trunk, there must be at least one vertex with degree at least 3 that would disconnect 2 colored leaves from the trunk if it were removed. Since this vertex can have at most 3 colored neighbors, there is at least one legal color that Alice can use to color this vertex.

Alice colors the aforementioned vertex with a legal color.

Position 2: It is Bob's turn.

Arrangement of the playing graph:

After Alice has colored this vertex, this trunk itself can be thought of a partially colored forest which can be decomposed into trunks with at most 2 colored leaves. Hence, the players are returned to position 0 (for any case described in this proof) and Alice is in a winning position since she has a winning strategy from position 0 in each case and sub case in this proof.

This is because Alice can color any vertex on the unique path between the two colored leaves since a vertex on this path can have at most 2 colored neighbors. If there is only a single colored leaf then Alice can start by coloring any vertex and still win the game.

Case 2: Position 0: Bob colors a vertex on a trunk in $\mathcal{R}$(F) that has less than 2 colored vertices.

(i) If there is only 1 colored vertex on the trunk on which Alice decides to play when its her turn, she can color any vertex to reach a winning position (since any vertex has at most one colored neighbor).

(ii) If there are 2 vertices colored on the trunk on which Alice decides to play when its her turn, she can begin by coloring any vertex on the unique path between the 2 vertices and reach a winning position (since any vertex on this path has at most two colored neighbors).

It is worth noting that Bob cannot be a hindrance to Alice because even if he ends up coloring a vertex Alice intended to, it is equivalent to if Alice colored the vertex herself. ■

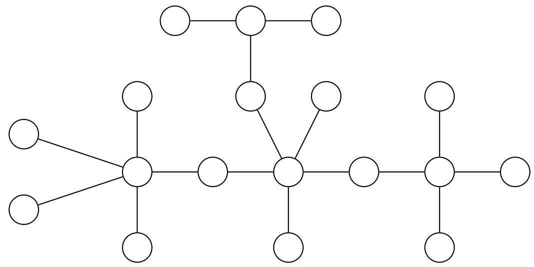
Now, we are ready to prove theorem 4.1,

Proof. Lemma 4.4 proved Alice can win a 4-MCG on $\mathcal{R}$(F). Hence, by lemma 4.2, Alice can win the 4-coloring game on any forest F. Hence,

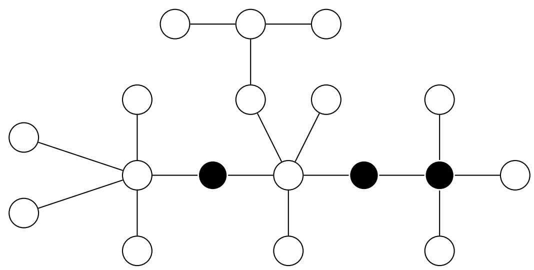
$$\chi_g(F) \leq 4$$

■

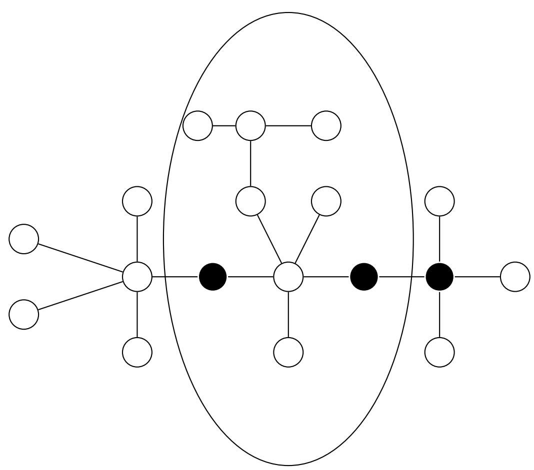
Let us understand this with an example. Consider the following forest F' as the playing graph of the simple vertex coloring game.

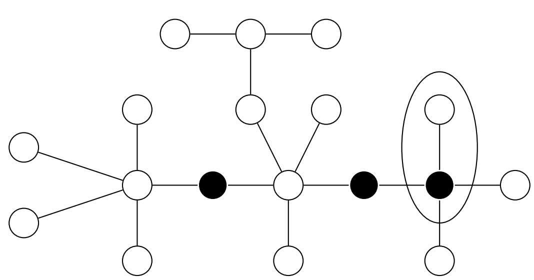


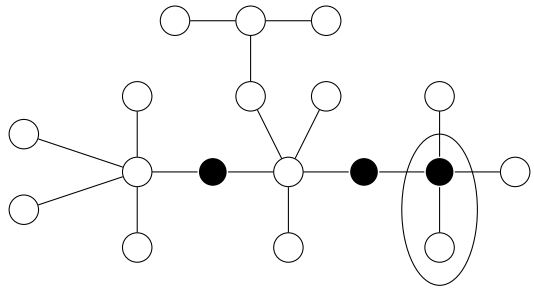
Let there be some partially colored version of this forest F' that shall be used as the playing graph of the Modified Coloring Game, shown below:

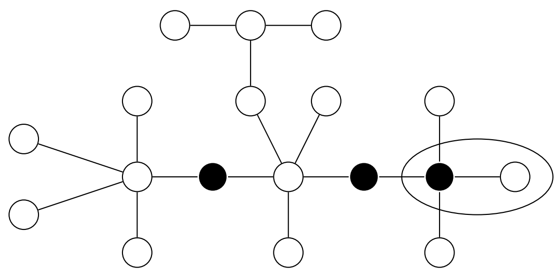


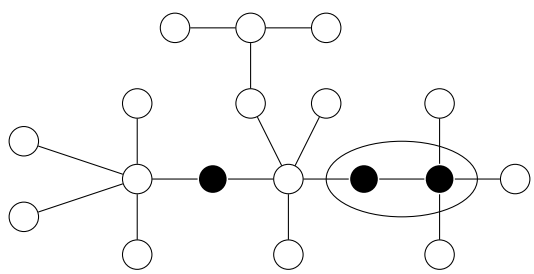
In the partially colored version of F', there are 6 trunks as circled below:

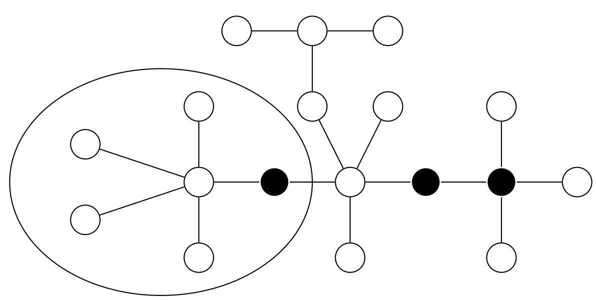






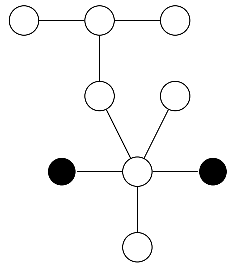






Hence, we can say $\mathcal{R}$(F') = 6. As we can see, each trunk has a maximum of 2 colored leaves. All uncolored vertices occur in only one trunk. There is a single trunk with 2 colored leaves and no uncolored vertices.

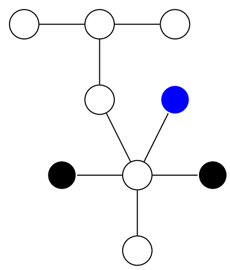
As in Case 1 in the proof of lemma 4.4, Let us assume Bob starts playing the MCG on $\mathcal{R}$(F') from the trunk shown below:



Let the palette consist of 4 colors black, blue, green and orange assigned the integer label 1,2,3 and 4 respectively. In his first turn, also the first turn of the game, Bob has 2 options:

- Color an uncolored leaf
- Color a vertex that is not a leaf

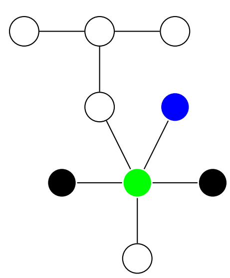
Let us assume Bob colors any uncolored leaf (the strategy we discuss subsequently is independent of Bob's choice of uncolored leaf). This is how the game would look after Bob's turn.



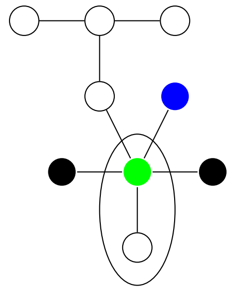
Note: although Bob could have colored the vertex black since the neighbor of the uncolored leaf is uncolored, Bob chooses blue in order to meet his goal. Recall, Bob's goal is to maximise the number of colors used for aesthetic purposes or to prevent the playing graph from having a proper coloring using the colors of the palette by the end of the game.

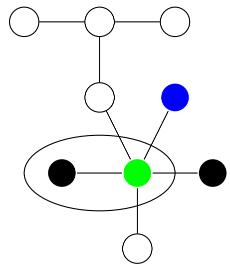
Now it is Alice's turn. The uncolored vertex which is not a leaf has a maximum of 3 colored neighbors. Hence, there are 2 legal colors for this vertex, namely green and orange or 3 and 4. Even if the 2 black vertices were differently colored, there would still be 1 legal color for this vertex. Hence, on this turn, Alice should color this vertex with a legal color, lets say green.

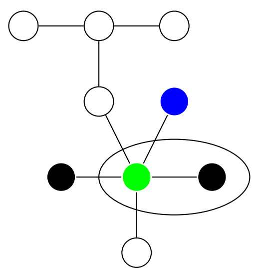
Now the playing graph looks like this:

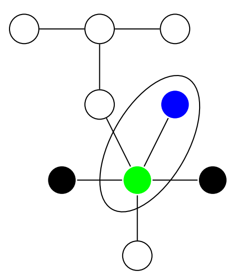


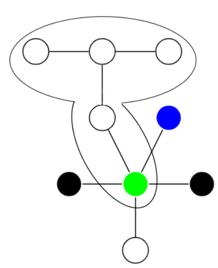
Now this trunk can be considered a partially colored forest in its own right and further decomposed into trunks as shown below:











In each of the trunks obtained like this, there are 3 possibilities:

- No uncolored vertex: Alice has already won
- One colored leaf: Alice is in a winning position
- Two uncolored leaves: Repeat strategy

Since Alice is in a winning position in all games on $\mathcal{R}$(F), Alice will win the simple vertex coloring on F.

5. Asymmetric Coloring Game on Undirected Forests

The following theorem is given without proof. The proof can be found in [Kie05].

Theorem 5.1. *Let a and b be positive integers. Then,*
(a) For $a < b$, $\chi_g(\mathcal{F}; a, b) = col_g(\mathcal{F}; a, b) = \infty$.
(b) For $b \leq a$, $b{+}2 \leq \chi_g(\mathcal{F}; a, b) \leq col_g(\mathcal{F}; a, b) \leq b{+}3$.
(c) For $b \leq a < max\{2b, 3\}$, $b{+}3 \leq \chi_g(\mathcal{F}; a, b)$.
(d) For $4 \leq 2b \leq a < 3b$, $\chi_g(\mathcal{F}; a, b) \leq b{+}2 < b{+}3 \leq col_g(\mathcal{F}; a, b)$

5.1. The Simple Vertex Coloring Game.

We can apply the above theorem to the simple vertex coloring game. As discussed earlier, the simple vertex coloring game is a special case of the (a,b)-coloring game where (a,b)=(1,1).

For the simple vertex coloring game:

- b $\leq$ a
- b $\leq$ a $< \{2b, 3\}$

Hence,

$$b + 2 \leq \chi_g(\mathcal{F}; a, b) \leq col_g(\mathcal{F}; a, b) \leq b + 3.$$

$$\implies 3 \leq \chi_g(\mathcal{F}; a, b) \leq col_g(\mathcal{F}; a, b) \leq 4$$

Also,

$$b + 3 \leq \chi_g(\mathcal{F}; a, b).$$

$$\implies 4 \leq \chi_g(\mathcal{F}; a, b).$$

The common solution to both these inequalities is:

$$\chi_g(\mathcal{F}; a, b) = 4.$$

or

$$\chi_g(\mathcal{F}; 1, 1) = 4.$$

This correlates with the result proved in the simple vertex coloring section.

6. Asymmetric Coloring Game on Oriented Forests

This section explores the (a,b)-coloring game on oriented forests. For oriented forests, the (a,b)-coloring game is modified slightly: instead of ensuring that all neighboring vertices differ in color from a given vertex, the players need only ensure that the in-neighbors of a vertex differ in color from that vertex. The inequality $\chi_g(D; a, b) \leq col_g(D; a, b)$ which holds for every digraph D.

Proposition 6.1. $\chi_g(D; a, b) \leq col_g(D; a, b)$

Proof. We begin with the assumption that there are col_g colors available to the players for a given graph, and that both Alice and Bob are playing with optimal strategies to achieve their goal (Alice's goal is to minimise the score obtained during the entire game while Bob's goal is to maximise the game's score).

The back degree of the linear order L of vertices in the game play is equal to the score. By definition, col_g is equal to the (back degree of L) + 1. Hence, there are as many colors as uncolored vertices and Alice simply needs to follow the optimal strategy to achieve her goal to choose her vertex each turn and color the vertex chosen with the smallest possible integer color available (following the first fit algorithm). The first-fit algorithm just assigns the least available and legal integer color to the given vertex.

Since this coloring assigns a different color to each vertex of the playing graph, it follows that the game chromatic number or the least number of colors in the palette needed for Alice to win can be at most equal to the number of vertices to color.

Hence, the inequality holds for any graph G and thus, for digraphs. When the game is asymmetric, Alice's play can be generalised to coloring a vertices in accordance with her optimal strategy with the least integer color available for each vertex. ∎

The above inequality will be used to prove the following theorem.

Theorem 6.2. *Let a and b be positive integers.Then,*
(a) for $b \leq a : \chi_g(\overrightarrow{\mathcal{F}}; a, b) = col_g(\overrightarrow{\mathcal{F}}; a, b) = b + 2$
(b) for $a < b : \chi_g(\overrightarrow{\mathcal{F}}; a, b) = col_g(\overrightarrow{\mathcal{F}}; a, b) = \infty$

The first part of the theorem gives the maximum χ_g attained by a graph belonging to the oriented forests class of graphs when the number of vertices B is required to color on each turn is less than or equal to the number of vertices A is required to color on each turn. Now we shall prove two lemmas that help establish that the upper and lower bounds for χ_g are equal and hence, prove the first part of this theorem.

Lemma 6.3. *For all a,* $b \in \mathbb{N}$*,* $b+2 \leq \chi_g(\overrightarrow{\mathcal{F}}; a, b)$.

Proof. Let there be an oriented forest F with vertices u_i, $v_{i,j}$ and $w_{i,j,k}$ such that

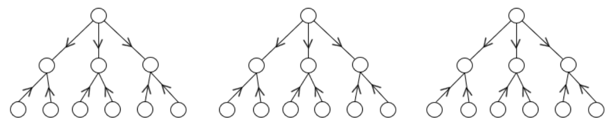
$$i = 1, 2, \ldots a + b$$

$$j = 1, 2, \ldots 2a + 1$$

$$k = 1, 2, \ldots b$$

The forest has arcs $(u_i, v_{i,j})$ and $(w_{i,j,k}, v_{i,j})$.

Let us take an example of such a forest when a=1 and b=2. The vertices of this forest are $u_1, u_2, u_3, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1}, v_{3,2}, v_{3,3}, w_{1,1,1}, w_{1,1,2}, w_{1,2,1}, w_{1,2,2}, w_{1,3,1}, w_{1,3,2}, w_{2,1,1}, w_{2,1,2}, w_{2,2,1}, w_{2,2,2}, w_{2,3,1}, w_{2,3,2}, w_{3,1,1}, w_{3,1,2}, w_{3,2,1}, w_{3,2,2}, w_{3,3,1},$ and $w_{3,3,2}$.
This is how such a forest would look:



Now, we shall prove that Bob always has a winning strategy for any number of colors $\leq$ b+1 by assuming b+1 colors are available in the palette.

Position 0: Initial Setup. Alice plays her turn.

Position 1: It is Bob's turn. No matter what Alice chooses to do, bob should color b of the vertices labelled u_i.

Position 2: Alice plays her turn.

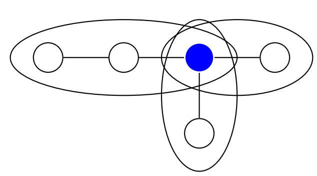
Position 3: There will be at least one sub tree left with the vertex set $\{u_{i_0}, v_{i_0,j_0}, w_{i_0,j_0,k} | k = 1, \ldots b.\}$. This implies that there is at least one vertex with b+1 in-neighbors. One in-neighbor is already colored. All that needs to be done is use the b colors remaining to color the rest.

Thus, Bob has won. Alice cannot color the vertex v_{i_0,j_0} in the above sub graph using b+1 colors. If there were lesser, Bob would have still won the game (as Bob would exhaust the distinct colors on the b+1 in-neighbors of v_{i_0,j_0}.
In the example taken, say i_0 is 1 and j_0 is 2. This vertex $v_{1,2}$ would have 3 in-neighbors. One in-neighbor u_1 is already colored. Hence, Bob on his turn colors the other 2 in-neighbors of the vertex with distinct colors. At most 3 distinct colors (b+1) are available to use in the palette which have already been exhausted. There is now no legal color left for Alice to use for $v_{1,2}$.

Hence proved. ∎

Lemma 6.4. *For all $b \leq a$, then $col_g(\overrightarrow{\mathcal{F}}; a, b) \leq b+2$*

The proof of this lemma requires us to understand the concept of independent sub trees. When a vertex of a sub tree of the playing graph forest is marked, we can regard it as splitting the sub tree into more sub trees where the marked vertex occurs in each of these smaller sub trees and is a leaf in them. If the in-degree of marked vertex is one, it can be disregarded, since it is not a threat to its neighbor according to the modified rules of the (a,b)-coloring game. Such sub trees are called independent sub trees. An example is shown below where the filled in vertex is marked and the independent sub trees are circled.



Proof. Let F be any oriented forest, the initial playing graph. It has been proven that Alice will always win the (1,1)-marking game with a score of 3 by guaranteeing that after each move every independent sub tree has at most one marked vertex. This strategy works even if Bob passes one or several turns.

We will now adapt this strategy such that the score of the (a,b)-marking game is b+2. In this new strategy, Alice still guarantees that after her moves every independent sub tree has at most one marked vertex. Then after Bob's next move every independent sub tree has at most b + 1 marked vertices.

Since Bob begins the game, at least 1 turn of Bob shall precede Alice's turn. On each turn, Bob marks b vertices. Say Bob has marked vertices $v_1, v_2, \ldots v_b$.

Now it is Alice's turn. She has to mark a vertices. She does this in steps:

(1) STEP 1: On the first step, Alice assumes Bob has only marked v_1 and proceeds to mark a vertex according to her winning strategy for the (1,1)-marking game unless she must mark some v_i where i belongs to 1,2,... b.

(2) STEP 2: On the second step, Alice assumes Bob has only marked v_1 and v_2. Then she proceeds to mark a vertex according to her winning strategy for the (1,1)-marking game unless she must mark some v_i where i belongs to 1,2,... b. In this case she marks the next vertex in her winning strategy.

(3) STEP k: On the k^{th} step, Alice assumes Bob has only marked $v_1, v_2, \ldots, v_k$. Then she proceeds to mark a vertex according to her winning strategy for the (1,1)-marking game unless she must mark some v_i where i belongs to 1,2,... b. In this case she marks the next vertex in her winning strategy.

(4) STEP x $\leq$ b: The graph invariant (E.g. chromatic number) is re-established (becomes the same as the original playing graph). *At most b+1 marked neighbours of any vertex.*

(5) STEPS a-x: Alice plays as if Bob was passing. She ensures that by the end of her turn, *each independent sub tree has at most one marked vertex.*

■

Proof. Now we prove the first part of this theorem. From the two lemmas proved above, it is established that b+2 $\leq \chi_g(\overrightarrow{\mathcal{F}}; a, b) \leq col_g(\overrightarrow{\mathcal{F}}; a, b) \leq$ b+2. Since, both the upper and lower bound are the same, it is proved that $\chi_g(\overrightarrow{\mathcal{F}}; a, b) = col_g(\overrightarrow{\mathcal{F}}; a, b) =$ b+2. ■

The second part of this theorem states that when the number of vertices Alice is required to color on her turn is strictly lesser than the number of vertices Bob is required to color, then χ_g becomes ∞ or Alice does not have a winning strategy for any number of colors in the palette. In other words, Bob always wins. The second part of this theorem is proved directly. If b > a, then $\chi_g(\overrightarrow{\mathcal{F}}; a, b) = \infty$

Proof. Let there be a positive integer c and an oriented forest F with vertex set

$$\{v_i, w_{i,j} | i = 1, 2, \ldots b^c, j = 1, 2, \ldots, c\}$$

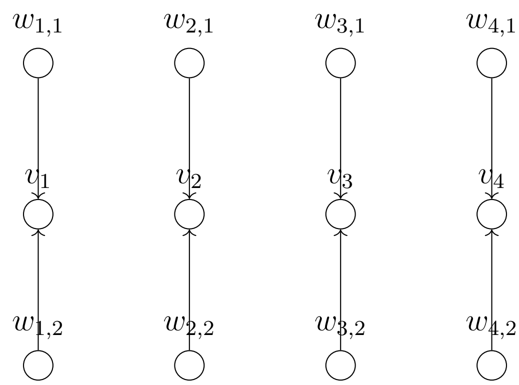
and an arc set of

$$\{(w_{i,j}, v_i | i = 1, 2, \ldots b^c, j = 1, 2, \ldots, c\}$$

.

For example, when a=1 and b=2, let c=2. i=1,2,3,4 and j=1,2.

The vertices of such a forest are $v_1, v_2, v_3, v_4, w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{1,2}, w_{2,2}, w_{3,2}$ and $w_{4,2}$. Such a graph would be as shown below:



Basically, F is a graph of b^c oriented stars.

Now, we will prove that Bob has a winning strategy for any c' $\leq$ c colors.

We assume that the game is divided into rounds where a single round consists of several turns of both players. Let k = $\{1, 2, \ldots c'\}$. During the k^{th} round, Bob colors with color given an integer label k, b^{c-k+1} vertices $w_{i,j}$ in stars where Alice has not colored any vertices previously and Bob has colored only vertices $w_{i,j}$ with colors 1,2,..., k-1. This is definitely possible since:

$$a \leq b$$

Since a and b are positive integers,

$$a \leq b - 1$$

Implies that,

$$b^{c-k}a \leq b^{c-k}(b-1)$$

In other words, Alice may color at most $b^{c-k+1} - b^{c-k}$ of the stars Bob has colored. Since there are b^c total stars, there are at least $b^{t-(k+1)+1}$ stars in the next or the $(k+1)^th$ round.

In the last c' round, there will be at least one star left with the center uncolored whose leaves are colored with c' colors. Thus, Bob wins.

Considering the example shown before:

c'=2

In the first round, Bob colors 4 vertices $w_{i,j}$ and 2,3 or 4 stars. Alice colors vertices in at most 2 of the stars Bob colors. If Bob colors 2 stars, the 2 stars are completely colored during this round.

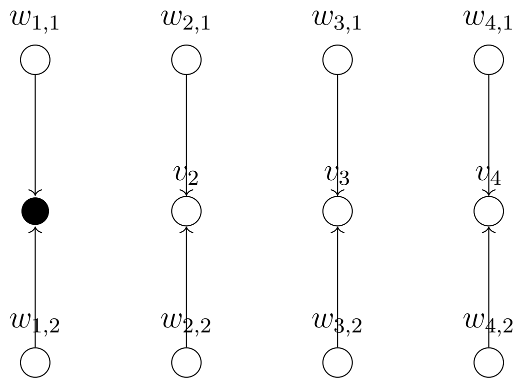
In round 2, Bob colors 2 vertices $w_{i,j}$ and Alice colors vertices in at most 1 star which Bob colors.

Since Alice ends up coloring vertices in at most 3 stars which Bob colors and Bob is only coloring vertices $w_{i,j}$, there is one star left with a vertex with 2 colored neighbors and no legal colors left.

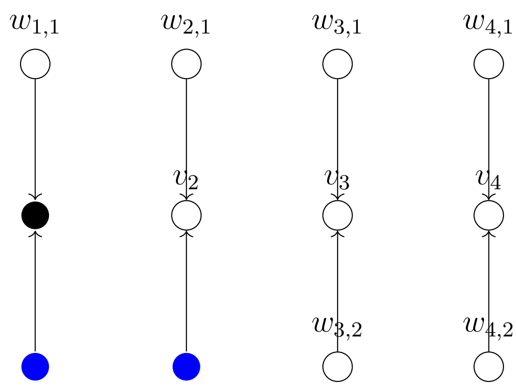
A Sample Game Play is as follows:

Round 1:

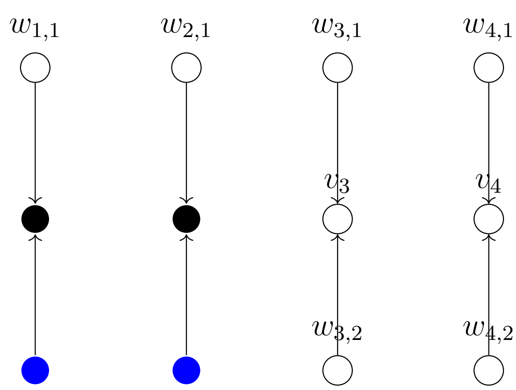
Position 1: Alice's play.



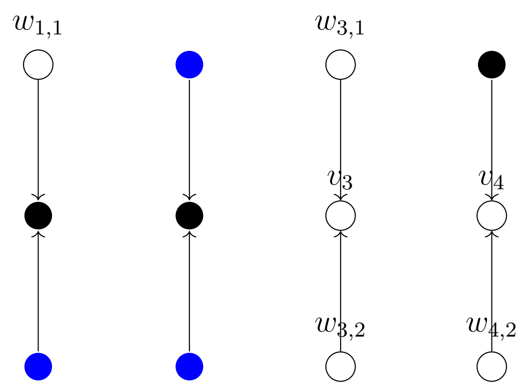
Position 2: Bob's play.



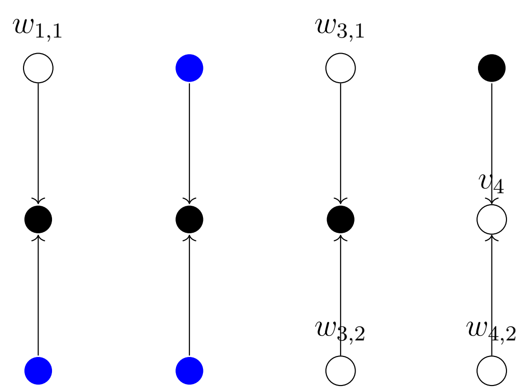
Position 3: Alice's play



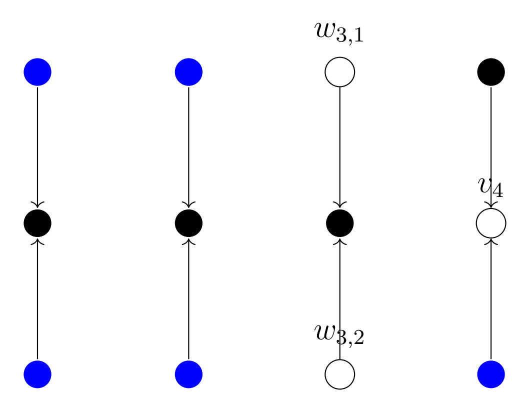
Position 4: Bob's Play



Round 2:
Position 5: Alice's Play



Position 6: Bob's Play:



Vertex v_4 does not have a legal color left. Hence, Bob wins this sample game.

This strategy can be applied by Bob for any integer c. Hence, there is no upper bound for c and Alice can never win. Hence proved. ∎

This completes the proof of theorem 6.2.

7. An Interesting Note on the Results for the Asymmetric Games

Let $\mathcal{H}$ be a class of graphs that is closed under taking disjoint unions and adding K as a component. It can be shown that $\chi_g(\mathcal{H})$ is independent of which player begins the game. Hence, the results are applicable even when Bob begins the game. Although it is worth

noting that the results for individual graphs belonging to the class may not be symmetric (or the same regardless of who begins the game).
Another interesting thing is that the game chromatic numbers of sub graphs of G are not bounded above by the game chromatic number of G.

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