The Galton-Watson Process

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What is the Galton-Watson Process?

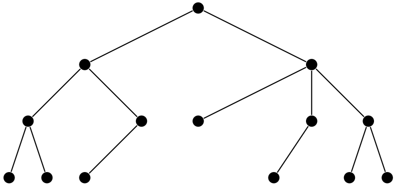
The Galton-Watson Process is a stochastic process which focuses on the evolution of a population size over time.

- The question appeared when many Victorians were concerned about aristocratic last names going extinct.
- Their solution is incomplete, stating that all family names go extinct with probability 1.

Origins

- Francis Galton saw this came up with a question, *"Determine the extinction rate of surnames and the number of surnames held by m individuals after r generations, given the distribution of male offspring reaching adulthood."*
- To this the Reverend Henry William Watson replied and both created a mathematical model for the propagation of family names.
- Many other mathematicians like Bienayme, Cournot, and Erlang tried proving this problem but died before any publications.

Branching Processes



$Z_0 = 1$

$Z_1 = 2$

$Z_2 = 5$

$Z_3 = 6$

The diagram above depicts a branching process where the Z_n denotes the number of individuals in the n-th generation. Assuming that $Z_0 = 1$ the sequence $Z_0, Z_1, ...$ is a branching process

The Galton-Watson Process Mathematically

- $$X_{n+1} = \sum_{j=1}^{X_n} \xi_j^{(n)}$$

- X: population size of a certain generation making X_{n+1} the size of the (n+1)-th generation.
- $\xi_j^{(n)}$ is the number of offspring produced by the j-th individual in the n-th generation

Extinction and cases

- μ stands for the average of a distribution
- intervals, a subcritical case, a critical case, and a supercritical case. The subcritical case is when $\mu < 1$, meaning that the average number of children per individual is less than one.
- Point two The critical case is when $\mu = 1$ so the average number of children per individual is exactly 1. The generation neither grows or decays but rather stays constant over time.
- Point three he supercritical case is exactly what it seems, the population size is $\mu > 1$.

Example

Consider a population where each individual has either 0, 1, or 2 offspring with probabilities $p_0 = 0.3$, $p_1 = 0.4$, and $p_2 = 0.3$, respectively. Starting with one individual in generation 0, calculate the expected population size in generation 2 and determine the probability that the population becomes extinct after two generations.

- $$\mu = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2$$
- $$\mu = 0 \cdot 0.3 + 1 \cdot 0.4 + 2 \cdot 0.3 = 1$$
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Generating function

A generating function is just another way to write a sequence.

- $$G(s) = \sum_{k=0}^{\infty} p_k s^k$$
- Extinction : $q = G(q)$

Example Cont.

Given the probabilities the generating function becomes.

- $$G(s) = p_0 s^0 + p_1 s^1 + p_2 s^2$$
- $$G(s) = 0.3 + 0.4s + 0.3s^2$$
- $$q = 0.3 + 0.4q + 0.3q^2$$

Example Cont.

Subtracting q from both sides and combining like terms .

- $$0.3q^2 - 0.6q + 0.3 = 0$$
- $$q^2 - 2q + 1 = 0$$
- $$(q-1)^2 = 0$$
- $$q = 1$$