Sets with the Property $D(n)$

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Euler Circle

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Introduction

Definition

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Greek mathematician Diophantus of Alexandria was the first to study and solve this problem (his definition had rationals)

$$\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}.$$

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Fermat used d as the next number in the set. Using that $d + 1$, $3d + 1$, and $8d + 1$ are all perfect squares, he got the *Diophantine quadruple*

$$\{1, 3, 8, 120\}.$$

Introduction

It was found that the *Diophantine quadruple* $\{1, 3, 8, 120\}$ could not be extended anymore.

Theorem

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There exists infinitely many Diophantine quadruples.

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Example

A triple from the quadruple we showed earlier, $\{1, 8, 120\}$ is a $D(1)$ triple. However, it's also a $D(721)$ triple. Meaning, $1 \times 8 + 721$, $1 \times 120 + 721$, and $8 \times 120 + 721$ are all perfect squares.

n as a Perfect Square

Theorem

There exists infinitely $D(n)$ quadruples when n is a perfect square.

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Proof.

If we multiply the elements of the infinitely many $D(1)$ quadruples by an integer k, we get infinitely many $D(k^2)$ quadruples. ■

Values of n

There are many values for n in which it has been proved whether or not a quadruple can be formed with the property $D(n)$.

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Theorem

If $n \equiv 2 \pmod 4$, there exists no $D(n)$ quadruple for that value of n.

Values of n

Proof.

This was proved by claiming that $\{a_1, a_2, a_3, a_4\}$ is a quadruple with the property n. Then we know that the product of two distinct numbers in this set are either 2 or 3 $\pmod 4$. This means none of the numbers in the set can be 0 $\pmod 4$. Thus, all of the numbers are of either $1, 2,$ or 3 $\pmod 4$. However, there is a duplicate so it's contradictory. ∎

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Proposition

There exists no $D(n)$ quadruple for $n = -1$ or $n = -4$.

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Theorem

If $n \neq 4k + 2$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, there exists at least one $D(n)$ quadruple for n.

Getting S

Numbers not of the form $4k + 2$ can be written as:

$$4k + 3, 8k + 1, 8k + 5, 8k, 16k + 4, 16k + 12.$$

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We have a starting positive integer a, we can make two $D(2a(2k+1)+1)$ quadruples:

$$\{a, a(3k+1)^2+2k, a(3k+2)^2+2k+2, 9a(2k+1)^2+8k+4\} \text{ and}$$
$$\{a, a(k+1)^2-2k, a(2k+1)^2-8k-4, ak^2-2k-2\}.$$

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$$
\begin{aligned}
D(4k+3) &: \{1, 9k^2+8k+1, 9k^2+14k+6, 36k^2+44k+13\} \\
D(8k+1) &: \{4, 9k^2-5, 9k^2+7k+2, 36k^2+4k\} \\
D(8k+5) &: \{2, 18k^2+14k+2, 18k^2+26k+10, 72k^2+80k+22\} \\
D(8k) &: \{1, 9k^2-8k, 9k^2-2k+1, 36k^2-20k+1\} \\
D(16k+4) &: \{4, 9k^2-4k-1, 9k^2+8k+3, 36k^2+8k\} \\
D(16k+12) &: \{2, 18k^2+16k+2, 18k^2+28k+12, 72k^2+88k+26\}
\end{aligned}
$$

Getting S

Some sets have equal elements or non-positives.

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$$-12, -7, -4, -3, -1, 0, 1, 3, 4, 5, 8, 9, 12, \text{ and } 20$$

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Remove 0, 1, 4, and 9

$$D(-12) : \{1, 12, 28, 76\}$$
$$D(-7) : \{1, 8, 11, 16\}$$

Set S

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Theorem

If $n \neq 4k + 2$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, there exists at least one $D(n)$ quadruple for n.

Conjecture

There exists no $D(n)$ quadruples when $n \in S$.

Getting T

Now let's use the second of the $D(2a(2k+1)+1)$ quadruples:

$$\{a, a(3k+1)^2+2k, a(3k+2)^2+2k+2, 9a(2k+1)^2+8k+4\} \text{ and}$$
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$$\{a, a(k+1)^2-2k, a(2k+1)^2-8k-4, ak^2-2k-2\}.$$

$$\begin{aligned}
D(4k+3) &: \{1, k^2-2k-2, k^2+1, 4k^2-4k-3\} \\
D(8k+1) &: \{4, k2-3k, k^2+k+2, 4k^2-4k\} \\
D(8k+5) &: \{2, 2k^2-2k-2, 2k^2+2k+2, 8k^2-2\} \\
D(8k) &: \{1, k^2-6k+1, k^2-4k+4, 4k^2-20k+9\} \\
D(16k+4) &: \{4, k^2-4k-1, k^2+3, 4k^2-8k\} \\
D(16k+12) &: \{2, 2k^2-4k-4, 2k^2+2, 8k^2-8k-6\}
\end{aligned}$$

Set T

Removing perfect squares as well as 11, 17, 33, and 40, we get

$$T = \{-15, -12, -7, 7, 13, 15, 21, 24, 28, 32, 48, 52, 60, 84\}$$

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Theorem

If $n \neq 4k+2$ and $n \notin S \cup T$, then there exists at least 2 unique $D(n)$ quadruples.

Size Estimate

Let's call our $D(n)$ set as G. Let's also define

$$M_n = \sup\{|G| : G \text{ is a set with the property } D(n)\}$$

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Example

$M_1 = 4$, meaning that The largest amount of elements a $D(1)$ set can contain is four.

Splitting into Groups

Large Elements:

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Very Small Elements:

$$C_n = \sup\{|G \cap [1, n^2]| : G \text{ is a set with the property } D(n)\}$$

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Lemma

$A_n \leq 21$ *for all nonzero integers* n.

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$B_n < 0.6114 \log |n| + 2.158$ *when* $|n| \leq 400$, *and*
$B_n < 0.6071 \log |n| + 2.152$ *when* $|n| > 400$.

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Lemma

$C_n < 11.006 \log|n|$ *for* $|n| > 400$.

Upper Bound

Theorem

$$M_n \leq 31 \ \textit{for}\ |n| \leq 400,$$
$$M_n < 15.476 \log n \ \textit{for}\ |n| > 400.$$

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Conjecture

$M_n \leq 6$ *for all nonzero* n.

Fibonacci and Lucas

Quadruple with the property $D(F_x^2)$ in which $x \in \mathbb{N}$:

$$\{2F_{x-1}, 2F_{x+1}, 2F_x^3 F_{x+1} F_{x+2}, 2F_{x+1} F_{x+2} F_{x+3}(2F_{x+1}^2 - F_x^2)\}$$

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Quadruple with the property $D(L_x^2)$:

$$\{F_{x-3}F_{x-2}F_{x+1}, F_{x-1}F_{x+2}F_{x+3}, F_x L_x^2, 4F_{x-1}^2 F_x F_{x+1}^2 (2F_{x-1}F_{x+1} - F_x^2)\}$$