ON SETS WITH THE PROPERTY $D(n)$

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Abstract. Diophantine m-tuples are sets of m distinct positive integers with the property that the product two distinct elements of the set increased by 1 is a perfect square. Sets with the property $D(n)$ follow the generalization that rather than adding 1, we add n to obtain a perfect square. This paper covers the results of the results and conclusions which can be drawn from the sets with the property of $D(n)$ for the various n's we can have

1. Introduction

The first person to study the problem of the product of two numbers increased by 1 forming a perfect square was the ancient Greek mathematician Diophantus of Alexandria. Diophantus was able to solve this problem with the set of four rational numbers

$$\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}.$$

However, this paper will only be regarding integers rather than the full field of rational numbers.

Definition 1.1. A *Diophantine m-tuple* is a set of m distinct positive integers with the property that the product of any two distinct elements of the set increased by 1 is a perfect square.

The first set of four integers found that confined to this property is the set

$$\{1, 3, 8, 120\}.$$

This quadruple was found by a French jurist and mathematician, Pierre de Fermat. To find this quadruple, he began with the *Diophantine triple*, $\{1, 3, 8\}$. Fermat put d as the fourth integer for the set so that it would still follow the property of *Diophantine m-tuples*. He then used the fact that $d+1$, $3d+1$, and $8d+1$ were all perfect squares to find that d being 120 worked. However, he couldn't find a fifth number to be added to this tuple. It was conjectured for a while that no *Diophantine quintuple* exists. Eventually it was proved that no *Diophantine quintuples* exist [12]. Also, it has been proved that there exists an infinite number of *Diophantine quadruples*. This paper won't include the proofs of these since it's not the focus. Instead, this paper focuses on $D(n)$-*tuples*.

Definition 1.2. A $D(n)$-*tuple*, in which n is an integer, is a set of distinct nonzero integers such that the product of any two distinct numbers in the set increased by n forms a perfect square.

The quadruple we just took a look at, $\{1, 3, 8, 120\}$, is a $D(1)$-*tuple* since that's what Definitions 1.1 implies.

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Example. We also have $\{-1,3,4\}$ which is a $D(4)$-*tuple*. This is because $-1\times 3+4=1^2$, $-1\times 4+4=0^2$, and $3\times 4+4=4^2$.

We will also be using the following two definitions of *Regular Diophantine m-tuples*.

Definition 1.3. A Diophantine triple, $\{a_1,a_2,a_3\}$ in which $a_1<a_2<a_3$, is *regular* if $a_3=a_1+a_2+2\sqrt{a_1a_2+1}$.

Definition 1.4. A Diophantine quadruple, $\{a_1,a_2,a_3,a_4\}$ in which $a_1<a_2<a_3<a_4$, is *regular* if $a_4=a_1+a_2+a_3+2a_1a_2a_3+2\sqrt{(a_1+1)(a_2+1)(a_3+1)}$.

It's conjectured that all Diophantine quadruples are *regular*, but a proof for that does not exist yet.

Since a large amount of time was dedicated to the findings on Diophantine quadruples which is the generic case in which $n=1$, the first question to come up is on which values of n there can and can't exist quadruples for. One of the first findings to notice is that when n is of the form $4k+2$ in which $k\in\mathbb{Z}$, there exists no quadruples. We can also find that there exists a $D(n)$ quadruple for values of n that aren't of the form $4k+2$ or are not in the set $S=\{-4,-3,-1,3,5,8,12,20\}$. However, out of these, it has only been proved for $n=-4$ and $n=-1$ that no $D(n)$ quadruple exists. We found S by using the $D(2a(2k+1)+1)$ quadruple

$$\{a,a(3k+1)^2+2k,a(3k+2)^2+2k+2,9a(2k+1)^2+8k+4\}.$$

We did this by manipulating a to get $D(n)$ quadruples for the remaining values of n (the values not of the form $4k+2$). Similarly, we can use a different $D(2a(2k+1)+1)$ quadruple

$$\{a,a(k+1)^2-2k,a(2k+1)^2-8k-4,ak^2-2k-2\}$$

to obtain a different set $T=\{-15,-12,-7,7,13,15,21,24,28,32,48,52,60,84\}$. From this, we are able to find that there exists at least two distinct $D(n)$ quadruples for values of $n\neq 4k+2$ and $n\notin S\cup T$.

We can use the fact that there are infinite Diophantine quadruples to show what happens for $D(n)$ when n is a perfect square other than the 1. This is done by multiplying the square root of that n to each element, proving

Theorem 1.5. *There exists an infinite number of $D(n)$ quadruples for all values of n which are perfect squares.*

Another difference between *Diophantine m-tuples* and *$D(n)$-tuples* is that for some values of n, there can exist sets with more than four elements that follow the properties of $D(n)$. Let's first define

$$M_n=\sup\{|G|: G \text{ is a set with the property } D(n)\}$$

To find an upper bound on the number of elements, it can be split into an upper bound on different groups of elements: large, small, and very small. We combine these three upper bounds to get

Theorem 1.6.

$$M_n\le 31 \text{ for } |n|\le 400,$$
$$M_n<15.476\log n \text{ for } |n|>400.$$

This is still a large upper bound and no $D(n)$ sets have been found with a size close to the upper bound for M_n. For prime numbers we can improve this upper bound for when the primes are very large, since the previous bound will be better for smaller primes. If we set p to be a prime number, we can find that $M_p < 3 \times 2^{168}$ and $M_{-p} < 3 \times 2^{168}$.

$D(n)$-tuples have a relation with the Fibonacci and Lucas numbers. One of the major findings is that a Diophantine quadruple can't consist of only Fibonacci numbers. However, we can still generate a Diophantine quadruple by using the Fibonacci numbers:

$$\{F_{2x}, F_{2x+2}, F_{2x+4}, 4F_{2x+1}F_{2x+2}F_{2x+3}\}$$

in which x is a whole number. Another relation ship between the Fibonacci and Lucas numbers is that

$$\{F_{2x}, F_{2x+8}, 9F_{2x+4}, 4F_{2x+2}F_{2x+4}F_{2x+6}\}$$

is a $D(F_x^2)$ quadruple, and

$$\{F_{x-3}F_{x-2}F_{x+1}, F_{x-1}F_{x+2}F_{x+3}, F_xL_x^2, 4F_{x-1}^2F_xF_{x+1}^2(2F_{x-1}F_{x+1} - F_x^2)\}.$$

is a $D(L_x^2)$ quadruple.

The upcoming section will talk about the restrictions on n in forming a quadruple, what values form quadruples, and the special cases. The size estimate section covers an upper bound for the number of elements in a set with the property $D(n)$ depending on n. The following section includes how a tuple can be a $D(n)$-tuple for multiple distinct n's. In the section after that includes a few of the many connections that these $D(n)$-tuples share with the Fibonacci and Lucas numbers.

2. Quadruples

For the generic Diophantine m-tuple in which only 1 is added to the product of the numbers from the set, it has been proved that the largest number of elements the set can contain is four [12], and there are infinitely many such quadruples. However, for the various n's that are added to the products instead of 1, there is a notable difference in which of these n's can form quadruples while which of them can only form triples.

We will begin by proving a useful

Theorem 2.1. *There exists infinitely many $D(n)$ quadruples for when n is a perfect square.*

Proof. To prove this we can use the fact that there exists infinitely many Diophantine quadruples. Since n is a perfect square, let's state $n = k^2$ where $k \in \mathbb{Z}$. Also, let $\{a_1, a_2, a_3, a_4\}$ be a Diophantine quadruple, so a $D(1)$ quadruple, in which $a \in \mathbb{Z}$. If we use $1 \leq i, j \leq 4$ for distinct i and j where $i \in \mathbb{N}$ and $j \in \mathbb{N}$, we can see that

$$a_ia_j + 1 = c_1^2$$

where $c \in Z$. If we multiply this entire equation by k^2, we get

$$k^2a_ia_j + k^2 = c_2^2.$$

We can rewrite $k^2a_ia_j$ as $ka_i \times ka_j$. This means that that the tuple $\{ka_i, ka_j\}$ follows the property of $D(k^2)$. Thus, proving that we can turn the infinitely many $D(1)$ quadruple into infinitely many $D(k^2)$ quadruples by multiplying all of the elements in the quadruple by k. ∎

2.1. **Finding What Values for n that Quadruples Exist.** The first case of n's which are not able to make quadruples is the simplest to prove.

Theorem 2.2. *If $n \equiv 2 \pmod 4$, there is no $D(n)$ quadruple.*

Proof. Suppose we have set $\{a_1, a_2, a_3, a_4\}$ as a $D(n)$ quadruple, and $1 \leq i, j \leq 4$ for distinct i and j where $i \in \mathbb{N}$ and $j \in \mathbb{N}$. Since

$$a_i a_j + n = c^2$$

where $c \in \mathbb{Z}$, we notice that

$$a_i a_j \equiv 2 \text{ or } 3 \pmod 4 \tag{2.1}$$

because $c^2 \equiv 0$ or $1 \pmod 4$. (2.1) implies that none of the numbers in the set can be $0 \pmod 4$, leaving the 4 numbers to be 1, 2, or $3 \pmod 4$. However, to be a quadruple, there would need to be a duplicate of either 1, 2, or $3 \pmod 4$. This contradicts (2.1), proving that there are no $D(n)$ quadruples if $n \equiv 2 \pmod 4$. ■

The case we accounted for is when $n = 4k + 2$ where $k \in \mathbb{Z}$, so we need to check what happens when n is of the form $4k$, $4k + 1$, or $4k + 3$. These numbers can be rewritten to be in the form $4k + 3$, $8k + 1$, $8k + 5$, $8k$, $16k + 4$, or $16k + 12$. However, in order to show the proof of these cases, we first need to show how a quadruple can be generated based on a starting integer.

Suppose we have a pair of integers, a_1, a_2, such that the pair follows the properties of $D(n)$:

$$a_1 a_2 + n = c^2. \tag{2.2}$$

We notice that we can include $a_1 + a_2 + 2c$ in our set and it would still follow the property of $D(n)$. As a quick verification, we can see that

$$a_1(a_1 + a_2 + 2c) + n = (a_1 + c)^2,$$
$$a_2(a_1 + a_2 + 2c) + n = (a_2 + c)^2.$$

Applying this same principle to the new pair, $a_2, a_1 + a_2 + 2c$, we are quickly able to obtain a fourth number for our set. The number added is $a_1 + 4a_2 + 4c$, and from that we get our $D(n)$ quadruple,

$$\{a_1, a_2, a_1 + a_2 + 2c, a_1 + 4a_2 + 4c\},$$

when the product of the first and last element increased by n is a perfect square.

If we insert $n = 2k$ into (2.2), we get

$$a_1 a_2 + 2k = c^2.$$

From here, we have that

$$a_1^2 + 4(c^2 - 2k) + 4a_1 c + n = h^2$$

where $h \in \mathbb{Z}$, and we have

$$(a_1 + 2c - h)(a_1 + 2c + h) = 6k.$$

We can split this into two separate equations to make

$$a_1 + 2c - h = 6,$$
$$a_1 + 2c + h = k.$$

From here, we can add both of these equations and multiply by 2 in order to get

$$2k = 4a_1 + 8c - 12.$$

Using what we originally inserted into (2.2), we are able to rewrite this as

$$a_1(a_2 + 2) = (c - 2)(c - 6).$$

When we replace $c = a_1k + 2$, we are able to find $2k$ as $4a_1(2k + 1) + 4$, enabling us to find the $D(4a_1(2k + 1) + 4)$ quadruple:

$$\{a_1, a_1k^2 - 4k - 4, a_2(1 + k)^2 - 4k, a_2(1 + 2k)^2 - 16k - 8\}. \tag{2.3}$$

Through manipulation of a_1, we are able to generate quadruples that fall under many multiples of k plus some constant. This formulates a base for how we can prove many $D(n)$ principles in which n is some mod k.

Theorem 2.3. *If n is an integer such that $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$ and $n \not\equiv 2 \pmod 4$, then there is at least one $D(n)$ quadruple.*

Of this set S, it has only been proved that no $D(n)$ quadruple exists for $n = -1$ [2] and $n = -4$. To show that no $D(-4)$ quadruple exists, it was proved that all elements of a $D(-4)$ quadruple must be even. We can make use of Theorem 2.1 by dividing all the elements of the $D(-4)$ quadruple by 2 to reach a $D(-1)$ quadruple. However, this is not possible since it was shown in [2] that no $D(-1)$ quadruple exists.

A conjecture exists the when n is an element in S, there doesn't exist a quadruple, but the remaining numbers are still yet to be proved. However, it has been proved that numbers not of S are able to form at least one $D(n)$ quadruple

Proof of Theorem 2.3. Similar to the quadruple (2.3) being a $D(4a(2k + 1) + 4)$ quadruple, it's proved [8, Chapter 3.6.1] that sets with the property $D(2a(2k+1)+1)$ can be expressed as a quadruple in the 2 following ways:

$$\{a, a(3k + 1)^2 + 2k, a(3k + 2)^2 + 2k + 2, 9a(2k + 1)^2 + 8k + 4\} \text{ and} \tag{2.4}$$

$$\{a, a(k + 1)^2 - 2k, a(2k + 1)^2 - 8k - 4, ak^2 - 2k - 2\}. \tag{2.5}$$

Using (2.4), a $D(2a(2k + 1) + 1)$ quadruple, we can manipulate $n = 2a(2k + 1) + 1$ by using a, making n be $4k+3$, $8k+1$, $8k+5$, $8k$, $16k+4$, and $16k+12$. $a = 1$ gives us a set with the property of $D(4k + 3)$. Setting $a = 4$ and substituting $k' = 2k + 1$, we get a quadruple with the property $D(8k' + 1)$. $a = 2$ gives us a set with the property of $D(8k + 5)$. Multiplying the elements of the set by 2, setting $a = \frac{1}{2}$, and substituting $k' = k + 1$, we get a quadruple with the property $D(8k')$. Setting $a = 2$ and substituting $k' = 2k + 1$ we get a quadruple with the property $(16k' + 4)$. Quadruples of the last property, $D(16k + 12)$, can be obtained by multiplying the elements of the quadruple for $D(4k + 3)$ by 2. From (2.4) and these manipulations, we have obtained the following quadruples with said properties:

$$\begin{aligned}
D(4k+3) &: \{1, 9k^2+8k+1, 9k^2+14k+6, 36k^2+44k+13\} \\
D(8k+1) &: \{4, 9k^2-5, 9k^2+7k+2, 36k^2+4k\} \\
D(8k+5) &: \{2, 18k^2+14k+2, 18k^2+26k+10, 72k^2+80k+22\} \\
D(8k) &: \{1, 9k^2-8k, 9k^2-2k+1, 36k^2-20k+1\} \\
D(16k+4) &: \{4, 9k^2-4k-1, 9k^2+8k+3, 36k^2+8k\} \\
D(16k+12) &: \{2, 18k^2+16k+2, 18k^2+28k+12, 72k^2+88k+26\}
\end{aligned}$$

Although it may seem like we have proved that all values for n not of the form $n = 4k+2$ are able to form a $D(n)$ quadruple, there is something that needs to be considered. If we insert small values as k into the quadruples, we run into the problem of elements in the set being equal or non-positive. $k = -1$ has this problem for the quadruples with properties $D(4k+3)$, $D(8k+1)$, $D(8k+5)$, $D(16k+4)$, and $D(16k+12)$. $k = 0$ has this problem for all 6 quadruples. $k = 1$ has this problem for quadruples with properties $D(8k+1)$, $D(8k)$, and $D(16+4)$. We obtain 14 numbers from inserting these values of k: -12, -7, -4, -3, -1, 0, 1, 3, 4, 5, 8, 9, 12, and 20. However, we have proved that there exists $D(n)$ quadruples for when n is a perfect square in Theorem 2.1, meaning we can remove 0, 1, 4, and 9 from our list in order to make the theorem more specific. It has also been proved that there exists a $D(-12)$ quadruple, $\{1, 12, 28, 76\}$, and a $D(-7)$ quadruple, $\{1, 8, 11, 16\}$. Since there exists a quadruple for these numbers, we are able to narrow down our list further to be $S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$. As stated earlier, a conjecture exists that the elements of S can't form a quadruple with the property $D(n)$. If proved true, then S can't be narrowed down further. Since this has not been proved yet, we stick to the theorem that if $n \notin S$ and $n \neq 4k+2$, then there exists at least one $D(n)$ quadruple. ■

Similarly, we can make sets of quadruples by using (2.5). The sets we can make also have property $D(2a(2k+1)+1)$, so we are able to manipulate them in the same way to form six quadruples with properties $D(4k+3)$, $D(8k+1)$, $D(8k+5)$, $D(8k)$, $D(16k+4)$, and $D(16k+12)$.

$$\begin{aligned}
D(4k+3) &: \{1, k^2-2k-2, k^2+1, 4k^2-4k-3\} \\
D(8k+1) &: \{4, k2-3k, k^2+k+2, 4k^2-4k\} \\
D(8k+5) &: \{2, 2k^2-2k-2, 2k^2+2k+2, 8k^2-2\} \\
D(8k) &: \{1, k^2-6k+1, k^2-4k+4, 4k^2-20k+9\} \\
D(16k+4) &: \{4, k^2-4k-1, k^2+3, 4k^2-8k\} \\
D(16k+12) &: \{2, 2k^2-4k-4, 2k^2+2, 8k^2-8k-6\}
\end{aligned}$$

Theorem 2.4. *Let $T = \{-15, -12, -7, 7, 13, 15, 21, 24, 28, 32, 48, 52, 60, 84\}$. If $n \neq 4k+2$ and $n \notin S \cup T$, then there exists at least 2 unique $D(n)$ quadruples.*

Similar to how we proved Theorem 2.3, we can also insert small numbers for k that run into the similar block, resulting in a list of numbers. We are able to narrow down the list by removing the perfect squares and by removing 11, 17, 33, and 40. We are able to remove these because of the quadruples from (2.4) and the fact that separate $D(n)$ quadruples can be made for these numbers: {2,7,19,35}, {1,8,19,208}, {8,51,101,296}, and {1,24,41,129} for

$D(11)$, $D(17)$, $D(33)$, and $D(40)$ respectively. Resulting from this, we get the narrowed down set of $T = \{-15, -12, -7, 7, 13, 15, 21, 24, 28, 32, 48, 52, 60, 84\}$. A much more formal demonstration of this can be found in [4].

2.2. **Specific Cases for Values of n.** For some values of n, there can be limitations on which types of numbers must be elements of the set in order to be able to from a quadruple. Similarly, some follow the principle of how specific types of numbers can't be part of a quadruple to form some $D(n)$ quadruple. An example of one of the types of numbers that follows a restriction on the elements of the set is $n \equiv 5 \pmod 8$.

Theorem 2.5. *If $n = 8k + 5$, all elements of a quadruple which follows property $D(n)$ must be $2 \pmod 4$.*

Proof. To do this we can show how there can't exist a $D(8k + 5)$ quadruple if any elements of the set are odd or are of $0 \pmod 4$. $\{a_1, a_2, a_3, a_4\}$ is our $D(8k + 5)$ quadruple, and $1 \leq i, j \leq 4$ for distinct i and j. Since $a_ia_j + n$ is a perfect square,

$$a_ia_j \equiv 3, 4, \text{ or } 7 \pmod 8.$$

If we assume a_1 to be even and a_2 to be odd, a_1a_2 must be $4 \pmod 8$, meaning $a_1 \equiv 4 \pmod 8$, and a_3 and a_4 must be odd. In order for a_1, a_2, a_3, and a_4 to form a quadruple,

$$a_2a_3, a_3a_4, \text{ and } a_4a_2 \equiv 3 \pmod 4,$$

but this is false since at least one of these products will be $1 \pmod 4$. This is because there are 3 odd numbers in the quadruple, there must be a duplicate of either $1 \pmod 4$ or $3 \pmod 4$. This proves 3 odd elements wouldn't work in a set with this property and neither will 4. Also, when we put a_1 as $4 \pmod 8$ which is $0 \pmod 4$, the remaining elements must be odd in order to avoid a product of $0 \pmod 8$, proving that there can't exist a $D(8k+5)$ quadruple if any elements are odd or $0 \pmod 4$. All elements of the quadruple must be of form $4k + 2$. ∎

Although it required a much more complicated process, the proof that there exists no $D(-1)$ quadruple initially started in a similar manner as how we proved Theorem 2.5. It involved many restrictions on what $D(-1)$ triples couldn't be extended, some of such are shown in [3].

For certain values of n, rather than there being a maximum for how many quadruples can be made with the property $D(n)$, there is a proved minimum for how many distinct $D(n)$ quadruples can be made. Namely, when n is of $1 \pmod 8$, $4 \pmod{32}$, or $0 \pmod{16}$, there exists at least six distinct $D(n)$ quadruples except for a few exceptions. The proof for all three of these is similar to one another, so this paper only includes the proof for $1 \pmod 8$, but the rest can be found in [6].

Theorem 2.6. *If $n = 8k + 1$ and $n \notin U = \{-15, -7, 17, 33\}$ then are at least six distinct quadruples with property $D(n)$.*

Proof. In a similar method to how we generated (2.3) by manipulating variables, the following sets represent the four quadruples that can be formed with the property of $D(8k+1)$.

$$\{4, 9k^2 - 5k, 9k^2 + 7k + 2, 36k^2 + 4k\}$$
$$\{4, k^2 - 3k, k^2 + k + 2, 4k^2 - 4k\}$$
$$\{8, \frac{1}{2}k(k+3) + 3, \frac{1}{2}k(k-5) + 1, 2k^2 - 2k\}$$
$$\{8, \frac{1}{2}k(9k-11) + 1, \frac{1}{2}k(9k+13) + 3, 18k^2 + 2k\}$$

In order to get two more quadruples to show how at least six distinct ones can be made, we can split $8k+1$ into $16k'+1$ and $16k'+9$ where $k' \in \mathbb{Z}$. The following sets show two quadruples that follow property $D(16k'+1)$.

$$\{k' - 3, 4k', 9k' - 1, 16k' - 8\}$$
$$\{4k', 25k' + 1, 49k' + 3, 144k' + 8\}$$

The following sets show two quadruples that follow property $D(16k'+9)$.

$$\{k', 16k' + 8, 25k' + 14, 36k' + 20\}$$
$$\{k' - 1, 4k', 9k' + 5, 16k' + 8\}$$

Although it might seem like it has been proved that there exists at least six distinct $D(n)$ quadruples when $n = 8k+1$, we still need to consider when elements are the same or sets coincide. We can eventually conclude that we can get six distinct $D(16k'+1)$ quadruples for $k' \notin \{-2, -1, 0, 1, 2, 3\}$ and six distinct $D(16k'+9)$ quadruples for $k' \notin \{-3, -1, 0, 1, 3\}$.

The resulting numbers we get for n are $-39, -31, -15, -7, 1, 9, 17, 25, 33, 49, 57$. However, we are able to narrow down this list. We can remove $1, 9,$ and 25 from the list because they are perfect squares. For $n = -31$, the quadruple generating sets were able to generate four distinct quadruples, with the other two being duplicates or not forming a quadruple. Still, we are able to form two quadruples different from what the sets generate, $\{1, 40, 47, 56\}$ and $\{1, 40, 287, 320\}$, proving that there are at least six quadruples with the property $D(-31)$. When $n = -39$ and $n = 57$, the sets were able to generate five distinct quadruples for each. Sixth $D(-39)$ and $D(57)$ quadruples that have not been account for are $\{1, 43, 48, 3520\}$ and $\{1, 7, 24, 232\}$ respectively. We were able to narrow down the list to be set $U = \{-15, -7, 17, 33\}$, proving that there exists at least six distinct $D(n)$ quadruples when $n = 8k+1$ and $n \notin U = \{-15, -7, 17, 33\}$. ■

As stated earlier, we can go through a very similar process to show how for each property $D(32k+4)$ and $D(16k)$, there exists at least six distinct quadruples with a few exceptions. The following two theorems have been proved in a similar manner [6]:

Theorem 2.7. *If $n = 32k+4$ and $n \notin V = \{-28, 68\}$ then are at least six distinct quadruples with property $D(n)$.*

Theorem 2.8. *If $n = 16k$ and $n \notin W = \{-16, -8, 8, 24, 32, 40, 48, 80\}$ then are at least six distinct quadruples with property $D(n)$.*

Another type of n that gives us a minimum for the number of distinct quadruples that can be made with the property $D(n)$ are numbers that are 8 (mod 16), 13 (mod 24), 21 (mod 24), 3 (mod 12), and 7 (mod 12) [6]. For these values of n, with the exception of a few numbers once again, there exists a minimum of four distinct $D(n)$ quadruples.

Theorem 2.9. *If $n = 24k+13$ or $24k+21$ and $n \notin X = -27, -11, -3, 13, 21, 45, 117$, there exists at least four distinct $D(n)$ quadruples.*

Theorem 2.10. *If $n = 12k+3$ or $12k+7$ and $n \notin Y = -9, -5, 3, 7, 15, 27, 63$, there exists at least four distinct $D(n)$ quadruples.*

3. Size Estimate

A common question that rises is how large can these $D(n)$-tuples get. Although it varies throughout the different values of n, we can form an upper bound on the supremum. It was found through forming cases based on the size of n. First, let's define

$$M_n = \sup\{|G| : G \text{ is a set with the property } D(n)\}$$

where $|G|$ refers to the number of elements in G. It has already been proved that M_4 and M_1 are 4. We also showed the proof of how $M_{4k+2} = 3$ and $M_n \geq 4$ for $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, but what reasonable size estimate can be formed for the rest of the values of n?

The proof began by forming a supremum for the number of elements that can be in the set for each of 3 cases: greater than $|n|^3$, between n^2 and $|n^3|$, and between 1 and n^2.

$$\begin{aligned} A_n &= \sup\{|G \cap [|n^3|, +\infty]| : G \text{ is a set with the property } D(n)\} \\ B_n &= \sup\{|G \cap (n^2, |n^3|)| : G \text{ is a set with the property } D(n)\} \\ C_n &= \sup\{|G \cap [1, n^2]| : G \text{ is a set with the property } D(n)\} \end{aligned}$$

Although it's evident that $M_n \leq A_n + B_n + C_n$ when all elements of the set are positive, it's different when there are mixed signs: $M_n \leq 2C_n$ [8, Chapter 5.2]. This is because if there are non-positives, then all the elements of the set must be of less than $|n|$ for a $D(n)$-tuple to exist. Thus, we get that $M_n \leq 2C_n$ in that case. However, this will not be of use until the next subsection. This is because we are able to get a better upper bound than we would get from using $M_n \leq 2C_n$. So, we have

$$M_n \leq \max(A_n + B_n + C_n, 2C_n) \tag{3.1}$$

From finding A_n, B_n, and C_n over intervals for n since some of the suprema vary depending on what interval n falls in, we are able to obtain three lemmas. Two of these lemmas rely on the gap principle, a detailed explanation on what it is can be found on [17]. The last of these lemmas uses the large sieve method.

Theorem 3.1.

$$\begin{aligned} &M_n \leq 31 \text{ for } |n| \leq 400, \\ &M_n < 15.476 \log n \text{ for } |n| > 400. \end{aligned}$$

Proof. A_n is the estimate of the number of large elements that can be in G. Through using a gap principle [8, Chapter 5.2.1], we can prove

Lemma 3.2. *$A_n \leq 21$ for all nonzero integers n.*

B_n is the estimate of the number of small elements that can be in G. By using a variant of the gap principle [7], we can prove

$$B_n < 0.65 \log |n| + 2.24.$$

Eventually, a more detailed analysis of the gap principle was done [8, Chapter 5.2.2], proving

Lemma 3.3. *$B_n < 0.6114 \log |n| + 2.158$ when $|n| \leq 400$, and $B_n < 0.6071 \log |n| + 2.152$ when $|n| > 400$.*

To find C_n, the estimate of the number of very small elements that can be in G, we have to go through a much more complicated process. It relies on using the large sieve method, finite fields fields obtained from sums, and other principles [8, Chapter 5.2.3] to obtain

Lemma 3.4. *$C_n < 11.006 \log |n|$ for $|n| > 400$.*

When $|n| \leq 400$, $C_n \leq 5$. This is easy to show through the use of a computer. Also, from combining that with Lemma 3.3 and Lemma 3.2, we are able to show the first part of Theorem 3.1:

$$M_n \leq 31 \text{ for } |n| \leq 400.$$

When $|n| > 400$, we are able to combine all three lemmas, proving the second part of Theorem 3.1:

$$M_n < 15.476 \log n \text{ for } |n| > 400.$$

■

These two inequalities form an upper bound for the supremum of how many elements exist in G. Still, farther reduction is most likely possible to make a much more sound estimate for M_n through the various values of n. It's conjectured that $M_n \leq 6$, a large distance from the upper bound. However, that result is far away form being proved and still just a conjecture.

3.1. **Primes.** For sets with the property of $D(p)$ and $D(-p)$ in which p is a prime number, the process to find the upper bound for M_p and M_{-p} is done quire differently. Let's say we have a set $P = \{a_1, a_2, \ldots, a_m\}$ and $1 \leq i, j \leq k$ in which $i \neq j$. We have that $a_i a_j + p = c^2$. When the tuple has property $D(-p)$, there can't be mixed signs. However, for $D(p)$, when there can be mixed signs, so we will just have to multiply the bound by 2 to get the absolute upper bound as we can see from (3.1) how this is done to C_n. Also, it is important to note that in the set P, there can only exist one multiple of p. We can show this through seeing what would happen if there were two multiples of p: a_i and a_j. We could say $a_i a_j + p = c^2$ since a_i and a_j follow the property of $D(p)$, but performing modulo p^2 results in

$$p \equiv c^2 \pmod{p^2}.$$

However, this is contradictory and wouldn't work, proving that there can be at most one multiple of p in set P.

Without loss of generality, let's assume that P is in increasing order. That is $0 < a_1 < a_2 < \cdots < a_m$. Also, if we have elements of the set, a_i and a_j, let's put the result of the product increased by p as $a_i a_j + p = c_{i,j}^2$ to allow c to assume multiple numbers. Immediately, we can begin by proving that $a_3 > p^{\frac{1}{4}}$. To do this, we can first note $a_1 a_3 + p = c_{1,3}^2$ and $a_2 a_3 + p = c_{2,3}^2$. Now we have

$$c_{2,3} > c_{1,3} > p^{\frac{1}{2}}.$$

Also,

$$a_3^2 > a_3(a_2 - a_1) = c_{2,3}^2 - c_{1,3}^2 = (c_{2,3} - c_{1,3})(c_{2,3} + c_{1,3}) = 2p^{\frac{1}{2}},$$

proving

$$a_3 > p^{\frac{1}{4}}.$$

If instead, it were a $D(-p)$-tuple,

$$a_2^2 > a_1 a_2 = c_{1,2}^2 + p > p, \text{ so}$$
$$a_2 < p^{\frac{1}{2}}.$$

In this case, the inequality from the $D(p)$ case still holds true. Also, if we eliminate the first 2 elements of the set, a_1, a_2, if needed, we are able to state that $a > p^{\frac{1}{4}}$ for all a in the set. In addition, we also can see that from Lemma 3.2, the number of elements in the set such that $a > p^3$ is less than or equal to 21. If we were to remove those elements from the set, we would have that for all a in set P, $p^{\frac{1}{4}} < a < p^3$. Through the use of this and the gap principle, the theorem below has been proved [9].

Theorem 3.5. *For a set $D(p)$, $M_p < 3 \times 2^{168}$, and for a set $D(-p)$, $M_{-p} < 3 \times 2^{168}$ for all prime numbers p.*

Also, for when $p \leq 2^{2^{76}}$, we can use Theorem 3.1 to notice that $M_p \leq 2^{80}$. This means that Theorem 3.5 is best used for when $p > 2^{2^{76}}$ since we can get a better bound for the other case.

4. $D(n)$ Sets With the Same Elements for Several n's

So far through this paper, $D(n)$ sets have only been looked at as though they were just a set of elements that follow the property of $D(n)$. However, in a new perspective, these sets can follow more than just the property of $D(n)$. The can be sets of $D(n_1)$, $D(n_2)$, $D(n_3)$, and so on. For example, we can show that such triples exist by using a part of the famous quadruple, $\{1, 3, 8, 120\}$, mentioned earlier. We already know the triple $\{1, 8, 120\}$ is a $D(1)$ triple. However, it's also a $D(721)$ triple. Another such triple is $\{8, 21, 55\}$, which is both a $D(1)$ and $D(4321)$ triple. Both of these immediately dismiss the initial conjecture of how if there is a $D(1)$ triple, that same triple can't follow the property of $D(n)$ where $n \neq 1$. Now, a question that rises is what is the maximum numbers that n could be such that a $D(1)$ triple follows the property of $D(n)$-tuples for all such n's.

Proposition 4.1. *For a triple $\{a_1, a_2, a_3\}$, it can only be a $D(n)$ triple for a finite number of distinct n's.*

Proof. In this set $\{a_1, a_2, a_3\}$ that is a $D(n)$ triple, if we replace the n being added for x, we are able to use the fact that

$$a_1 a_2 + x = c_{1,2}^2$$
$$a_2 a_3 + x = c_{2,3}^2$$
$$a_3 a_1 + x = c_{1,3}^2$$

in which $c \in \mathbb{Z}$. From this, we are able to generate the equation

$$y^2 = (a_1 a_2 + x)(a_2 a_3 + x)(a_3 a_1 + x).$$

We are able to replace all the c^2's multiplied together to be y^2. This equation is an elliptic curve, and these curves are closely studied when studying on Diophantine m-tuples. It also helps in partially deducing an answer to our initial question of how many distinct values there can be for n such that the triple $\{a_1, a_2, a_3\}$ is a $D(n)$ triple for all of these values of n. Since this elliptic curve has a finite number of integer points, we know that a triple can't be a $D(n)$ triple for an infinite number of values for n. An elementary proof of this explanation can be found on [19] ■

Also, another question that rose was on how many triples there are such that are $D(1)$, $D(n_1)$, and $D(n_2)$ triples. It has been proved there exists infinite different triples that can be expressed as $D(1)$, $D(n_1)$, and $D(n_2)$ in which n_2 and $n_3 \neq 1$ and $n_1 \neq n_2$. Specifically, what was proved was

Theorem 4.2. *For the $D(1)$ quadruple $\{2, a_1, a_2, a_3\}$, the triple $\{a_1, a_2, a_3\}$ forms a $D(1)$ triple as well as a $D(n)$ triple for two distinct values of n and $n \neq 1$.*

To prove this we will be using

Lemma 4.3. *For the positive integers a_1, a_2, and a_3 where $a_1 + a_2 + a_3$ is even, $\{a_1, a_2, a_3\}$ is a $D(n)$ set where*

$$n_1 = \frac{1}{4}(a_1 + a_2 + a_3)^2 - a_1a_2 - a_2a_3 - a_3a_1$$

given that $n_1 \neq 0$.

In this theorem, $n_1 = 1$ only when $a_3 = a_1 + a_2 \pm 2\sqrt{a_1a_2 + 1}$. Also, $n_1 \neq 0$, so we are able to obtain

Corollary 4.4. *A $D(1)$ triple, $\{a_1, a_2, a_3\}$, in which $a_1 + a_2 + a_3$ is even and $a_3 \neq a_1 + a_2 \pm 2\sqrt{a_1a_2 + 1}$, it's a triple that follows the property of $D(1)$ and $D(n_1)$ for $n_1 \neq 1$.*

We can analyze $2a_1 + 1$, $2a_2 + 1$, and $2a_3 + 1$ to notice that since all of them must result in a perfect square, we can conclude that a_1,a_2, and a_3 are all even due to the property of perfect squares being either 0 or 1 (mod 4).

Using Corollary 4.4, we are able to find one of such distinct n's. To find another distinct n such that $n_2 \neq 1$ and n_2 is not the same as the n_1 found by using Corollary 4.4, we have to go through a different process. From [1], we can find that it's proved that $n_2 = a_1 + a_2 + a_3$ with the except of some values of n. We are able to get this in [1] because $\{2, a_1, a_2, a_3\}$ is a regular Diophantine quadruple. More formally, we have

Lemma 4.5. *Let $\{2, a_1, a_2, a_3\}$ be a Diophantine quadruple. Then $\{a_1, a_2, a_3\}$ is a $D(n)$ triple for $n = a_1 + a_2 + a_3$.*

Without loss of generality we can say that the quadruple $\{a_1, a_2, a_3\}$ is in increasing order. We have that $2 < a_1 < a_2 < a_3$, so we know that $a_1 + a_2 + a_3 \geq 12$. Lemma 4.5 was proved [1] to not work for when $n \in \{-5, -3, -2, 0, 1, 3\}$, but we know that $n \geq 12$ so we can disregard this. Also, from this we know that $n_2 \neq 1$.

To check that $n_1 \neq n_2$ we are able to see that if $n_1 = n_2$ would lead to a contradiction when $a_3 \geq 1024$. However, when $a_3 < 1024$, the only triple $\{a_1, a_2, a_3\}$ that we can get where where $\{2, a_1, a_2, a_3\}$ is a regular Diophantine quadruple we can get is $\{4, 12, 420\}$. In this case, $n_1 \neq n_2$, thus we have proved Theorem 4.2.

Moving on from triples, the new question is whether or not there exist a set $\{a_1, a_2, a_3, a_4\}$ that is a $D(n)$ quadruple for more than 1 distinct values of n. One demonstration of this showing that it's certainly possible is the set $\{-1, 7, 64, 119\}$ being both a $D(128)$ and $D(848)$ quadruple. In fact, it was proved in [10] that there exists infinitely many quadruples such that the quadruple is a $D(n)$ quadruple for two distinct nonzero values of n.

5. Relation with Fibonacci Numbers

These $D(n)$-tuples also have connections with the Fibonacci and Lucas numbers. The Fibonacci numbers being F_n such that $F_1 = 1$, $F_2 = 1$, and $F_{x+2} = F_{x+1} + F_x$ in which x is a whole number. The Lucas numbers can be expressed as L_n such that $L_1 = 1$, $L_2 = 3$, and $L_{x+2} = L_{x+1} + L_x$. The Lucas numbers can also be written as $L_x = F_{x-1} + F_{x+1}$.

The first question that commonly rises when people learn that a connection between Fibonacci numbers and $D(n)$-tuples exists is whether or not there exists a $D(n)$ quadruple of Fibonacci numbers. For triples, there exists infinitely many. Any triple that falls under the form

$$\{F_{2x}, F_{2x+2}, F_{2x+4}\}$$

forms a $D(1)$ triple. The 4th term can be created by multiplying terms of the Fibonacci numbers together. Many have also proved

Theorem 5.1. *No regular Diophantine quadruples exist consisting of only Fibonacci numbers.*

To prove this, we need multiple lemmas and equations which will be listed bellow.

Lemma 5.2. *Let's say that x and k are positive integers, and $\{F_{2x}, F_{2x+2}, F_k\}$ is a Diophantine triple. Then $k = 2x + 4$ or $k = 2x - 2$ (only when $x > 1$) except for $x = 2$, in which case $k = 1$ would work.*

We will also be using the Binet formula for F_n in which we will set $(\alpha, \beta) = (\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ to get

$$F_x = \frac{\alpha^x - \beta^x}{\alpha - \beta} \text{ when } x \geq 0. \tag{5.1}$$

Also, from the use of the Binet formula, we can get an estimate on the size of F_x through using the inequality

$$\alpha^{x-2} \leq F_x \leq \alpha^{x-1} \text{ when } x \geq 1. \tag{5.2}$$

Similarly, the Lucas numbers also have a Binet formula being

$$L_x = \alpha^x + \beta^x \text{ when } x \geq 0. \tag{5.3}$$

The Fibonacci number and Lucas numbers have many equations to relate the two. One which will be useful for us to make our proof is

$$L_x^2 - 5F_x^2 = 4(-1)^x \text{ when } x \geq 0. \tag{5.4}$$

The equations and inequalities listed above can be found in [18]. We will also be using

Lemma 5.3. *If $F_xF_{x+2} + 1$ or $F_xF_{x+4} + 1$ is a perfect square for when $x > 2$, x must be even.*

Proof. To show this, we have to start by using the fact that when x is odd, $F_xF_{x+2}-1 = F_{x+1}^2$ and $F_xF_{x+4}-1 = F_{x+2}^2$. If x were the sane odd and $F_xF_{x+2}+1$ and $F_xF_{x+4}+1$ were perfect squares, it would mean the difference between the perfect squares would be 2, which wouldn't make sense proving that x must be even. ■

Another useful lemma to us is

Lemma 5.4. *We have $k \in \{1, 3\}$. If $F_xF_{x+k}+1$ is a perfect square for positive integers x, then $x = 4$ and $k = 1$.*

Proof. We can begin by using formulas (5.1) and (5.3) to get

$$F_xF_{x+k}+1 = \frac{1}{5}(\alpha^x - \beta^x)(\alpha^{x+k} - \beta^{x+k}) + 1 = \frac{1}{5}(L_{2x+k} - (-1)^x L_k + 5).$$

This means that for $F_xF_{x+k}+1 = t^2$ for $t \in \mathbb{Z}$, so for $F_xF_{x+k}+1$ to be a perfect square,

$$L_{2x+k} = 5t^2 + ((-1)^x L_k - 5).$$

If we insert this into formula (5.4) and replace x with $2x+k$, we get

$$5F_{2x+k}^2 = L_{2x+k}^2 - 4(-1)^k = 25t^4 + 10((-1)^x L_k - 5)^2 - 4(-1)^k. \tag{5.5}$$

Let's first assume that $k = 1$. We notice that if x were even or if x were odd we would be left with two separate cases. Also, let's set $y = F_{2x+k}$. In the case of x being even, we are able to reduce (5.5) to $y^2 = 5t^4 - 8t^2 + 4$. From here it's possible to find that the two positive integer solutions for (t, y) are $(1, 1)$ and $(4, 34)$. Since $y = F_{2x+1}$, it follows that for the solutions of y, 1 and 34, we have that $x = 0$ and 4 respectively. Alternatively, when x is odd, we can reduce (5.5) to $y^2 = 5t^4 - 12t^2 + 8$. This equation only has one positive integer solution which is $y = 1$ and $t = 1$. Once again, if y is 1 and also $y = F_{2x+1}$, then $x = 0$. However, both of the $x = 0$ solutions aren't convenient for us since this proof is for positive integers x which is all that's useful for us in proving Theorem 5.1. This means the only solution for x we are left with is $x = 4$

Now let's assume that $k = 3$. In the case which x is even, we have $5y^2 = 25t^4 - 20t^2 + 8$. In this case, there are no positive integer solutions. Similarly, there are no positive integer solutions to $5y^2 = 25t^4 - 80t^2 + 68$, proving that for $k \in \{1, 3\}$, $F_xF_{x+k}+1$ is a perfect square only when $k = 1$ and $x = 4$. ■

We also will be using

Lemma 5.5. *If $F_x + 1$ is a perfect square for positive integers x, then $x \in \{4, 6\}$.*

Proof. If we say that $F_x + 1 = t^2$ for positive integers t, we can rewrite this as $F_x = t^2 - 1$. If we insert this into (5.4) and also set $y = L_x$, we get

$$y^2 = 5F_x^2 + 4(-1)^x = 5t^4 - 10t^2 + 5 + 4(-1)^x$$

Similar to how we proved Lemma 5.4, we will also split this into two cases, x be even and x being odd. We then can get the equations, $y^2 = 5t^4 - 10t^2 + 9$ and $y^2 = 5t^4 - 10t^2 + 1$ for even and odd x respectively. The positive integer solutions to the equation we get from x being even are $(t, y) = (1, 2), (2, 7), (3, 18)$. We get no positive integer solutions when x is odd. Now that we have the t values of 1, 2, and 3, we can insert this back into $F_x = t^-1$. From this, we get $F_x = 0, 3, 8$ for $t = 1, 2, 3$ respectively. Also, when $F_x = 0$, $x = 0$ which is not a positive integer and also not helpful in proving Theorem 5.1. Thus, we have proved that if $F_x + 1$ to be a perfect square for positive integers x, then $x \in \{1, 3\}$. ■

Theorem 5.6. *If we have a regular Diophantine triple $\{F_{x_1}, F_{x_2}, F_{x_3}\}$ such that $x_1 < x_2 < x_3$, then we have that $x_1 + 2 = x_2$ and $x_1 + 4 = x_3$.*

Proof. We have that $F_{x_3} = F_{x_1} + F_{x_2} + 2\sqrt{F_{x_1}F_{x_2} + 1}$ (recall Definition 1.3). From here, we can get that $F_{x_2} < F_{x_3} < 4F_{x_2} < F_{x_2} + 4$. From this inequality, we can see that because $F_{x_3} < F_{x_2} + 4$, we can get that $F_{x_3} = F_{x_2} + c$ for some $c \in \{1, 2, 3\}$. From here, we can use that $F_{x_2}F_{x_3} + 1$ is a perfect square. For now, if we say that $c \neq 2$, then $c \in \{1, 3\}$. Using Lemma 5.4, it would mean that $x_1 = 4$ and $x_3 = 5$, so $F_{x_1} = 3$ and $F_{x_1} = 5$. If we put this into the triple, we would have $\{F_{x_1}, 3, 5\}$. However, neither $F_{x_1} = 1$ nor $F_{x_1} = 2$ work, thus proving that we must have $c = 2$. Also, from Lemma 5.3, we can see that x_2 must be even. Using Lemma 5.2 and due to x_2 being even, it satisfies the property conditions, thus proving that $x_1 = x_2 - 2$. There is a contradiction listed in Lemma 5.2 which states that when $x_2 = 4$, in addition to $x_1 = 2$, we can also have $x_1 = 1$. Still, Theorem 5.6 holds true, thus proving it. ■

Proof of Theorem 5.1. Suppose we have a *regular* Diophantine quadruple, $\{F_{x_1}, F_{x_2}.F_{x_3}, F_{x_4}\}$ (recall Definition 1.4). We will set it in order by stating $x_1 < x_2 < x_3 < x_4$. Also we know that if an element of this set was 1 meaning $x_1 = 1$ or 2 it wouldn't be able to form a *regular* quadruple consisting of Fibonacci numbers. This is since x_2 and x_3 could only be 4 or 6 as we can see from Lemma 5.5, so we wouldn't be able to get an x_4 to satisfy the conditions. Thus, F_{x_1} bust be greater than 1, meaning $x_1 \geq 3$.

Now, from using Lemma 5.2 as well as Lemma 5.4 we can see that $x_2 \geq x_1 + 4 \geq 7$ and $x_3 \geq x_2 + 4 \geq 11$. Also, we can find a lower bound for x_4 since

$$F_{x_3}(4F_{x_1}F_{x_2} + 1) < F_{x_4} < 4F_{x_3}(F_{x_1}F_{x_2} + 1), \tag{5.6}$$

we can get that $x_4 \geq x_1 + x_2 + x_3 - 2 \geq 19$. Another result we get from (5.6) is that

$$\frac{4}{5}(1 - \frac{1}{\alpha^{2x_3}})(1 + \frac{1}{\alpha^{2x_4}})^{-1}((1 - \frac{1}{\alpha^{2x_1}})(1 - \frac{1}{\alpha^{2x_2}}) + \frac{5}{4}\alpha^{-x_1-x_2}) < \alpha^{x_4-x_3-x_2-x_1} \text{ and}$$
$$\alpha^{x_4-x_3-x_2-x_1} < \frac{4}{5}(1 + \frac{1}{\alpha^{2x_3}})(1 - \frac{1}{\alpha^{2x_4}})^{-1}((1 + \frac{1}{\alpha^{2x_1}})(1 + \frac{1}{\alpha^{2x_2}}) + \frac{5}{4}\alpha^{-x_1-x_2}).$$

This means that $0.76 < \alpha^{x_4-x_3-x_2-x_1} < 0.88$. However, this is not possible. We can see this by showing that for for when $x_4 - x_3 - x_2 - x_1$ is a positive integer, it wouldn't work since $\alpha^{x_4-x_3-x_2-x_1} > 0.88$. Also for α^{-1} and any integer smaller for the exponent, the result will be less than 0.76. Thus, we are unable to form a *regular* Diophantine quadruple consisting of only Fibonacci numbers. ■

After it was proved that *regular* Diophantine quadruples could not consist of only Fibonacci numbers, it was also eventually proved that *irregular* Diophantine quadruples could not consist of only Fibonacci numbers [11]. Thus, through using both of these, they were able to form

Theorem 5.7. *There exists no four positive integers x_1, x_2, x_3, x_4 such that $\{F_{x_1}, F_{x_2}, F_{x_3}, F_{x_4}\}$ is a Diophantine quadruple.*

One of the earlier demonstrations of how the Fibonacci numbers are connected is done through Diophantine m-tuples, the generic case of $D(1)$. It was discovered that the set

$\{F_{2x}, F_{2x+2}, F_{2x+4}, 4F_{2x+1}F_{2x+2}F_{2x+3}\}$ is a $D(1)$ quadruple [13]. A simple test shows how so.

$$F_{2x} \times F_{2x+2} + 1 = F_{2x+1}^2$$
$$F_{2x} \times F_{2x+4} + 1 = F_{2x+2}^2$$
$$F_{2x} \times 4F_{2x+1}F_{2x+2}F_{2x+3} + 1 = (2F_{2x+1}F_{2x+2} - 1)^2$$
$$F_{2x+2} \times F_{2x+4} + 1 = F_{2x+3}^2$$
$$F_{2x+2} \times 4F_{2x+1}F_{2x+2}F_{2x+3} + 1 = (2F_{2x+2}^2 + 1)^2$$
$$F_{2x+4} \times 4F_{2x+1}F_{2x+2}F_{2x+3} + 1 = (2F_{2x+2}F_{2x+3} + 1)^2$$

Now let's demonstrate how the Fibonacci numbers correlate with with more than just $D(1)$. We can use a, such that $a \in \mathbb{Z}$, to generate 2 numbers for our quadruple to have, $k - a$, and $k + a$, such that

$$(k - a)(k + a) + a^2 = y^2 \tag{5.7}$$

where $y \in \mathbb{Z}$. We can insert $a = 1$ into (5.7). Through this, we can obtain other elements to form a quadruple.

$$\{k - 1, k + 1, 4k, 16k^3 - 4k\},$$
$$\{k - 1, k + 1, 16k^3 - 4k, 64k^5 - 48k^3 + 8k\}$$

are examples of such quadruples that can be obtained for $D(1)$. If we insert $a = 2$ into (5.7), we can get quadruples

$$\{k - 2, k + 2, 4k, 4k^3 - 4k\},$$
$$\{k - 2, k + 2, 4k^3 - 4k, 4k^5 - 12k^3 + 8k\},$$

and more quadruples that follow the same pattern in order to obtain $D(4)$ quadruples. Also, $\{k - 4, k + 4, 4k, 4k^3 - 4k\}$ is a $D(16)$ quadruple for $k > 4$. This demonstrates how various $D(a^2)$ quadruples are able to form various $D(a^2)$ quadruples by using this principle.

Now, we can use a Fibonacci relation [18] to show that

$$F_{2i}F_{2i+2j} + F_j^2 = F_{2i+j}^2.$$

There is a similarity to this and (5.7). We can substitute $a = F_j$. The first two elements of the set that will be used are F_{2i} and F_{2i+2j}. The k from the equation reflects F_{2i+j}. By using $j = 1, 2, 3, 4, 6$, and by using the Lucas number property of $L_x = F_{x-1} + F_{x+1}$, the following four theorems have been reached.

Theorem 5.8. *The following sets are quadruples that have the property $D(1)$.*

$$\{F_{2x}, F_{2x+2}, F_{2x+4}, 4F_{2x+1}F_{2x+2}F_{2x+3}\}$$
$$\{F_{2x}, F_{2x+4}, 5F_{2x+2}, 4L_{2x+1}F_{2x+2}L_{2x+3}\}$$

Theorem 5.9. *The following sets are quadruples that have the property $D(4)$.*

$$\{F_{2x}, F_{2x+6}, 4F_{2x+2}, 4F_{2x+1}F_{2x+3}F_{2x+4}\}$$
$$\{F_{2x}, F_{2x+6}, 4F_{2x+4}, 4F_{2x+2}F_{2x+3}F_{2x+5}\}$$

Theorem 5.10. *The following set is a quadruple that has the property $D(9)$.*

$$\{F_{2x}, F_{2x+8}, 9F_{2x+4}, 4F_{2x+2}F_{2x+4}F_{2x+6}\}$$

Theorem 5.11. *The following set is a quadruple that has the property* $D(64)$.

$$\{F_{2x}, F_{2x+12}, 16F_{2x+6}, F_{2x+3}F_{2x+6}F_{2x+9}\}$$

Another connection between the two is when $n = F_x^2$ or when $n = L_x^2$. Through using the identities

$$4F_{x-1}F_{x+1} + F_x^2 = L_x^2,$$
$$(F_x^2 + F_{x-1}F_{x+1})^2 - 4F_{x-1}F_{x+1}F_x^2 = 1,$$

it's been proved in [5] that we are able to obtain a quadruple with the property $D(F_x^2)$:

$$\{2F_{x-1}, 2F_{x+1}, 2F_x^3F_{x+1}F_{x+2}, 2F_{x+1}F_{x+2}F_{x+3}(2F_{x+1}^2 - F_x^2)\}.$$

By using the Morgado identity [16]

$$F_{x-3}F_{x-2}F_{x-1}F_{x+1}F_{x+2}F_{x+3} + L_x^2 = (F_x(2F_{x-1}F_{x+1} - F_x^2))^2,$$

and by using the identities

$$4F_{x-2}F_{x+2} + L_x^2 = 9F_x^2,$$
$$(F_x^2 + F_{x-2}F_{x+2})^2 - 4F_{x-2}F_{x+2}F_x^2 = 1,$$

it's been proved in [5] that we are able to obtain a quadruple with the property $D(L_x^2)$:

$$\{F_{x-3}F_{x-2}F_{x+1}, F_{x-1}F_{x+2}F_{x+3}, F_xL_x^2, 4F_{x-1}^2F_xF_{x+1}^2(2F_{x-1}F_{x+1} - F_x^2)\}.$$

Another idea that has been considered and researched is that if Fibonacci numbers could be the result of 1 added to the product of two distinct numbers in the set. This idea steers away from the result being a perfect square, so it doesn't follow the property of $D(n)$. For example, suppose we have a $D(1)$ triple $\{a_1, a_2, a_3\}$. It has been proved in [14] that $a_1a_2 + 1$, $a_2a_3 + 1$, and $a_3a_1 + 1$ cannot all three result in a Fibonacci number. For Lucas numbers, it doesn't follow the same rule as the Fibonacci numbers, and instead, the following theorem has been proved in [15].

Proposition 5.12. *The set of positive integers* $\{a_1, a_2, a_3\}$ *such that*

$$a_1a_2 + 1 = L_{x_1}$$
$$a_2a_3 + 1 = L_{x_2}$$
$$a_3a_1 + 1 = L_{x_3}$$

for $a_1 < a_2 < a_3$ *only exists when* $\{a_1, a_2, a_3\} = \{1, 2, 3\}$

In this single existing case, $a_1a_2 + 1 = L_2 = 3$, $a_2a_3 + 1 = L_4 = 7$, and $a_3a_1 + 1 = L_3 = 4$.

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