Central Limit Theorem

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Random Variables

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- Real value outcome from an event, for example the outcome of rolling a fair 6-sided die
- Continuous random variables are assigned a value from a continuous range, while discrete random variables are assigned a value from a discrete range, which may or may not be infinitely large

Expected value

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- Calculated by summing up the product of each outcome's value and its probability

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \Pr(x) dx$$

or

$$\mathbb{E}[X] = \sum_{i} x_i \Pr(x_i)$$

Variance

For a random variable,

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Variance

Another useful form of variance can be derived from our definition for random variables:

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 \text{ by linearity of expectations} \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2\end{aligned}$$

Moments

- The kth moment of a random variable can be calculated with

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- Moments are a way of characterizing a distribution, to the point where if all moments of two distributions are equal, then the distributions are identical

Normal Distribution

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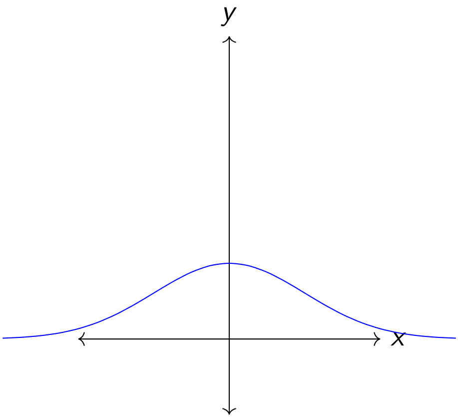
- All normal distributions have area 1 and follow the equation

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

- A special type of normal distribution, the standard normal distribution, has mean 0 and variance 1

Standard Normal Distribution



A Weak Central Limit Theorem

Central Limit Theorem

Given independent, identically distributed (i.i.d) random variables $X_1, X_2, \ldots, X_n$ with mean 0 and variance 1, as $n \to \infty$,

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \to \mathcal{N}(0, 1).$$

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Essentially, the distribution of a normalized sum of random variables will approach the standard normal distribution as $n \to \infty$.

A Proof Using Moments

The kth moment of a standard normal variable is

$$\mathbb{E}[Z^k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-\frac{1}{2}x^2} dx$$

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Using integration by parts, this can be simplified to $\mathbb{E}[Z^k] = (k-1)\mathbb{E}[Z^{k-2}]$.

A Proof Using Moments

Therefore,

$$\mathbb{E}[Z^k] = \begin{cases} (k-1)(k-3)\dots(2)\mathbb{E}[Z^1] & \text{if } k \text{ is odd,} \\ (k-1)(k-3)\dots(1)\mathbb{E}[Z^0] & \text{if } k \text{ is even.} \end{cases}$$

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We know that $\mathbb{E}[Z^1] = 0$ since Z has mean 0, and that $\mathbb{E}[Z^0] = \mathbb{E}[1] = 1$, so

$$\mathbb{E}[Z^k] = \begin{cases} (k-1)(k-3)\dots(2)(0) = 0 & \text{if } k \text{ is odd,} \\ (k-1)(k-3)\dots(1)(1) = (k-1)!! & \text{if } k \text{ is even.} \end{cases}$$

A Proof Using Moments (Greatly Summarized Part)

On the other hand, the kth moment of our sum is

$$\mathbb{E}[(X_1 + X_2 + \ldots + X_n)^k]$$

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When distributing and then splitting into individual expected values, we only want to care about terms that are comprised of squares of a random variable

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  - Since we normalize by dividing by $n^{\frac{k}{2}}$ on both sides, as $n \to \infty$, these terms approach 0
- That leaves terms with only squares of random variables, for example $\mathbb{E}[X_i^2 X_j^2]$

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The number of ways to pair partition in a group of k is $(k-1)!!$, so as $n \to \infty$,

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \to \mathcal{N}(0, 1).$$

Other Central Limit Theorems

- Other more applicable versions of the central limit theorem exist

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- For example, the Liapunov central limit theorem states that the convergence to a normal distribution also applies to independent and non-identically distributed variables under certain conditions

Some Results

Fair dice are independent and identically distributed random variables, so CLT will apply to them

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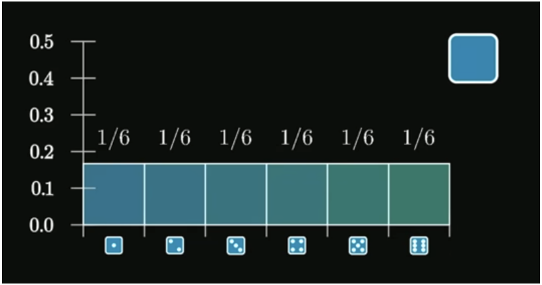

Figure 1: The fair die (n=1)

Some Results

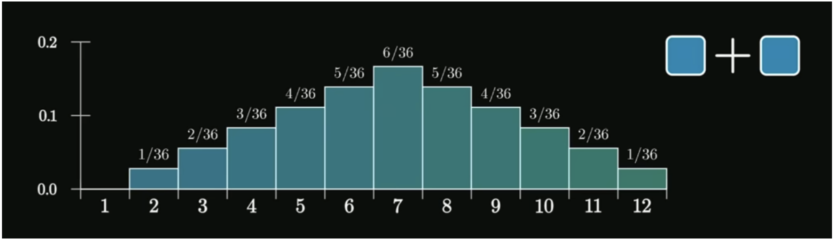

Figure 2: n=2

Some Results

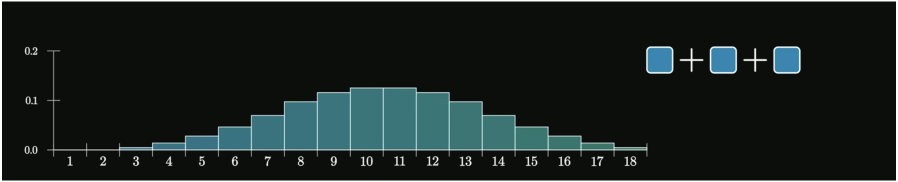

Figure 3: n=3

Some Results

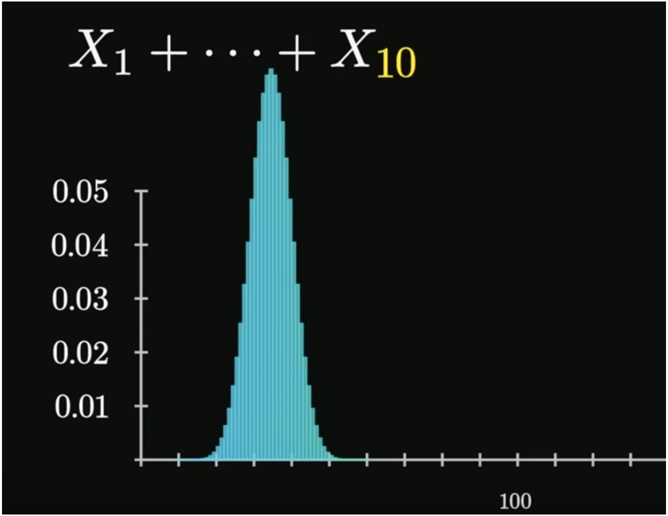

Figure 4: n=10

More Results

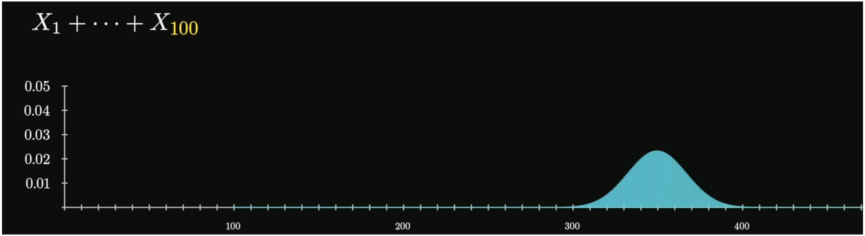

Figure 5: n=100

More Results

We may apply it to unfair dice too, as they are also independent and identically distributed random variables

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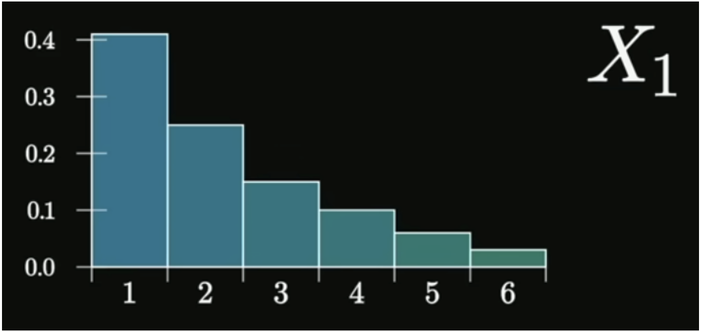

Figure 6: unfair die

More Results

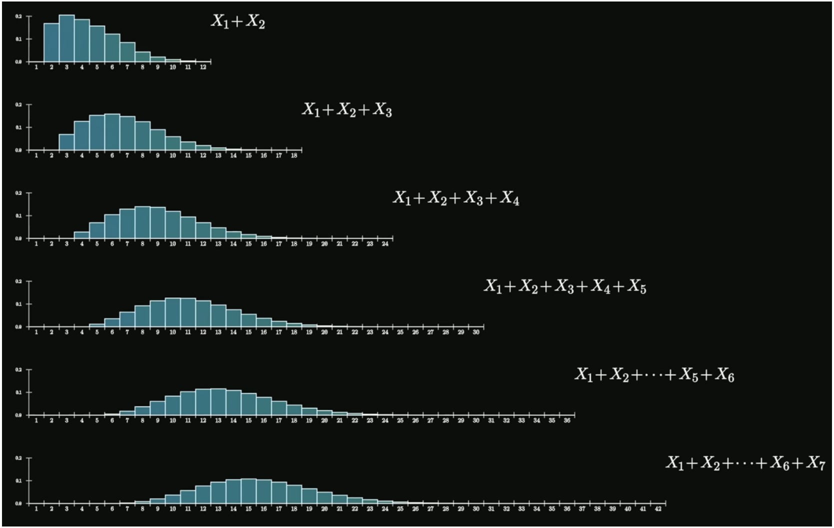

Figure 7: Convergence with unfair dice

More Results

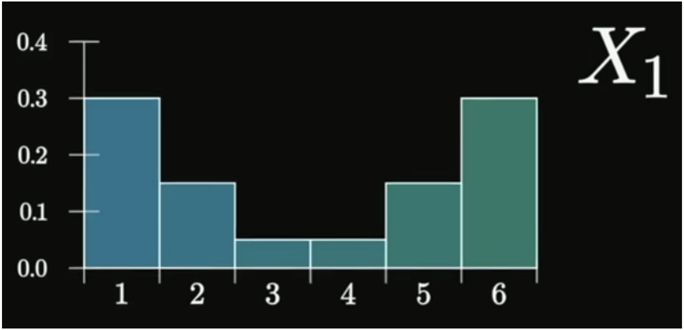

Figure 8: A different unfair die

More Results

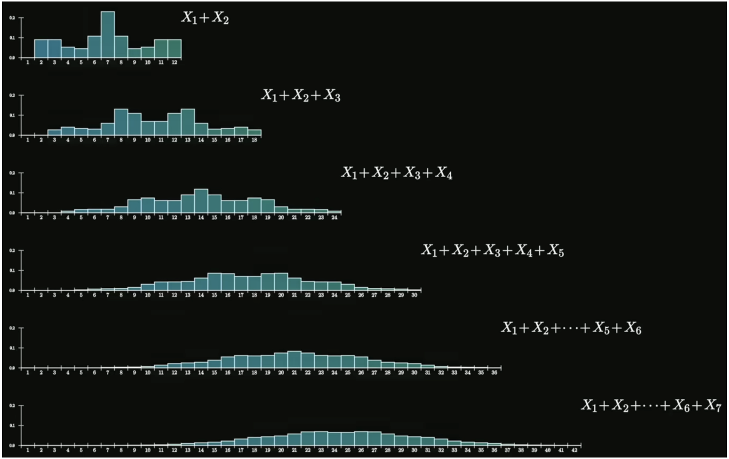

Figure 9: Convergence with different unfair dice

Conclusions (Why is this important?)

- The Central Limit Theorem is important because it allows statisticians to assume distributions of large sums are normal

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Conclusions (Why is this important?)

- The Central Limit Theorem is important because it allows statisticians to assume distributions of large sums are normal
- For example, if you were manufacturing bottles, when taking the sums of bottle volumes for many different groups of bottles, those sums will approximately fall into a normal distribution
- This allows the use of more statistical tools because you know the distribution is approximately normal

Thanks for listening!