Introducing the Theory of Repeated games

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July 12, 2024

Table of Contents

An introduction to games

The Prisoner's Dilemma

		Player 2	
		$cooperate(C)$	$defect(D)$
Player 1	$cooperate$	$1, 1$	$-1, 2$
	$defect$	$2, -1$	$0, 0$

Fig 1 - The Prisoner's Dilemma

What is a game?

Definition

The stage game, g, in a strategic form consists of a triplet $g = (I, (S_i)_{i \in I}, (U_i)_{i \in I})$, where

1. I is the set of players ($I = 1, \ldots, n$)
2. For each $i \in I$, S_i is a Player i's set of strategies, where $S = \prod_{i \in I} S_i$ is the set of strategy profiles.
3. For each $i \in I$, $U_i : S \to \mathbb{R}$ is the payoff function. A payoff function maps the n-tuple strategies s to real number values.

Important Notation

For this presentation, please note the following notation:

1. A stage game (the building block of a repeated game) is denoted by g.
2. A finitely repeated game is denoted by $G(T)$, where T is the number of times the stage game g is repeated.
3. An infinitely repeated game is denoted by $G(\infty)$.

Introducing repeated games

Finite Games vs Infinite Games

We define the two types of games as:

Definition

A Finitely Repeated game is a situation where players are aware of the number of times the stage game is repeated until the game ends.

Definition

An Infinitely Repeated game is a situation where players always believe that the game extends one more period with high probability.

Computing payoffs in Infinitely Repeated Games

The equation to calculate payoffs in an infinitely repeated game is as follows:

$$\sum_{t=0}^{\infty} \delta^t u_i \left(a^t \right) \tag{2.1}$$

Now, you may be wondering what δ denotes, it denotes the discount factor, which is exactly (in my opinion) what makes repeated games so interesting.

The Discount Factor

Definition

In a game, a discount factor is a value $0 \leq \delta \leq 1$ used to represent a player's pure time preference.

Looking at it in the context of the prisoner's dilemma:

1. A higher discount factor in an Infinitely Repeated Game means that players value future rewards almost as highly as present ones, and are hence more likely to cooperate.
2. A lower value of δ implies the player is more impatient, since the player puts less weight on payoffs in the future, relative to payoffs in the present, and are hence more likely to defect.

Introducing the Nash Equilibrium

What is a Nash Equilibrium?

Definition

A Nash equilibrium is a strategy profile s_i^* with the property that no player i can do better by choosing a strategy different from s_i^*, given that every other player $-i$ adheres to s_{-i}^*)
It is an action profile a_i^* $(a_i^* \in A_i)$ for player i, where, for all $i \in I$.

$$u_i\left(\ s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$$

The existence of a Nash Equilibrium in a stage game

Theorem

(Nash 1950) There exists a mixed Nash Equilibrium in all strategic form games.

While the proof is too lengthy to discuss in this talk, I will explain how to go about it.
We prove it using Kakutani's Fixed-point theorem.

Theorem

(Kakutani, 1938) In a non - empty, compact, and convex space, under certain conditions, there will be a point within this space that gets mapped by a perfect equilibriumcial function to a set containing itself.

Proving the existence of a Nash Equilibrium in a stage game

To apply Kakutani's fixed point theorem, 4 conditions need to be satisfied.
r is a player's reaction correspondence (which basically maps each strategy profile to the set of mixed strategies that maximize a player's conditional payoff), $r : S \to S$.

1. S is a compact, convex nonempty subset of a Euclidean space.
2. $r(s)$ is non-empty for all s.
3. $r(s)$ is convex for all s.
4. it has a closed graph (this is the same as upper - hemicontinuity)

Nash Equilibrium contd

When a game is in a nash equilibrium, every single player i plays it's best s_i in response to their opponents s_{-i}.

		Player 2: $cooperate(C)$	Player 2: $defect(D)$
Player 1	$cooperate$	$1, 1$	$-1, 2$
Player 1	$defect$	$2, -1$	$0, 0$

Observing the prisoner's dilemma, we can conclude that the best response for both players to any strategy is to defect, therefore the nash equilibrium in a game played only once is (D, D).

Subgame perfection

note: a subgame and a stagegame, in this presentation will be the same thing.

Definition

In a repeated game, a strategy profile is a subgame perfect equilibrium if it represents a Nash equilibrium of every subgame of the original game. This is also called a perfect equilibrium.

Remark

A nash equilibrium of the stage game and perfect equilibrium of the repeated game don't necessarily include the same strategies!

Perfect Equilibria

Grim Trigger Strategies

A Grim Trigger Strategy is a strategy in which player i will always cooperate until player $-i$ defects. It is represented by:

$$s_i\left(h^t\right) = \begin{cases} C_i & \text{if } t = 0 \text{ or } h^t = (C, C, \ldots, C) \\ D_i & \text{otherwise} \end{cases} \tag{4.1}$$

Existence of (C,C) Equilibria

We stated earlier that a nash equilibrium of the stage game and perfect equilibrium of the repeated game don't necessarily include the same strategy profiles. This is due to the value of the discount factor.

Proposition

In the infinitely repeated prisoners' dilemma, if $\delta \geq \frac{1}{2}$ there is an equilibrium in which (C, C) is played in every g

Proof of existence of (C, C) Equilibria

We now prove the proposition stated in the previous slide.

Proof.

To prove that there is no profitable single deviation, suppose D has already been played. Then, player i has two choices:

1. Play C for a payoff of $-1 + \delta \times 0 + \delta^2 \times 0 + \ldots = -1$ (referring to equation 4.4)
2. Play D for a payoff of $0 + \delta \times 0 + \delta^2 \times 0 + \ldots = 0$

Assuming that player i will want to maximise his payoff, player i should play D. Now, assume D has not been played, then, player i will have 2 choices:

1. Play C for a payoff of $1 + \delta + \delta^2 + \ldots = 1/(1 - \delta)$
2. Play D for a payoff of $2 + \delta 0 + \delta^2 \times 0 + \ldots = 2$.

Changing Equilibria with changing discount factors

Remark

We cannot say that people always cooperate when they interact. Cooperation is only one possible perfect equilibrium outcome. There are many others.

1. For any δ, there is a perfect equilibrium in which players play D in every period this is because (D, D) is the Nash Equilibrium.
2. For $\delta \geq \frac{1}{2}$, there is a perfect equilibrium in which the players play D in the first period and C in every following period.
3. For $\delta \geq \frac{1}{\sqrt{2}}$, there is a perfect equilibrium in which the players alternate between (C, C) and (D, D).

Introducing the Folk Theorem

The Basic Folk Theorem

- The folk theorem is a famous concept in repeated game theory.
- It applies when players are very patient and consider future interactions heavily (discount factor approaches 1).
- In such scenarios, repeated games can have many Subgame Perfect Equilibria (SPE) outcomes.
- Interestingly, the folk theorem suggests these equilibria can deliver virtually any average payoff outcome that is achievable in the single-stage game.

Axelrod's Tournament

Axelrod's Tournament

- In 1980, Robert Axelrod invited a number of well-known game theorists to submit strategies to be run by computers. In the tournament, programs played games against each other and themselves repeatedly. Each strategy specified whether to cooperate or defect based on the previous moves of both the strategy and its opponent.
- some of the strategies submitted included always defect, always cooperate, and random (where the strategy cooperated 50% of the time)
- The winner of Axelrod's tournament was the TIT FOR TAT strategy. The strategy cooperates on the first move, and then does whatever its opponent has done on the previous move.

Thank you!

Thank you for listening!