Nonstandard Methods in Infinite Ramsey's Theorem

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1 Introduction

This paper explores the application of nonstandard methods to the Infinite Ramsey's Theorem, highlighting how ultrafilters, nonstandard analysis, and hyperfinite generators contribute to our understanding of combinatorial structures in infinite graphs and hypergraphs. The discussion begins with an introduction to ultrafilters and their crucial role in nonstandard analysis, providing a foundation for analyzing infinite combinatorial systems. We then delve into the basics of nonstandard analysis, including the transfer principle and its implications for extending classical results.

The concept of hyperfinite generators is introduced to illustrate how finite structures can approximate infinite combinatorial configurations. We explore the use of many stars and iterated nonstandard extensions to construct and analyze large combinatorial objects. By employing these techniques, we demonstrate how infinite Ramsey's theorem can be effectively proven in the context of hypergraphs and general hypergraph theory.

The paper aims to bridge classical combinatorial results with modern nonstandard analytical methods, offering a comprehensive view of how these tools can be applied to solve problems in infinite Ramsey theory.[3]

2 Ultrafilters

Ultrafilters are a fundamental concept in topology and set theory, playing a crucial role in various branches of mathematics, including nonstandard analysis. They can be seen as a tool for selecting "large" subsets of a given set in a maximal way, and they have profound implications in understanding convergence, compactness, and other topological properties.

2.1 Definition and Basic Properties

Definition 2.1. Given an infinite set S, a filter $\mathcal{F}$ on S is a collection of subsets of S that satisfies the following properties:

1. $\emptyset \notin \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
3. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

We consider the elements of $\mathcal{F}$ as "large" sets since filters generally capture large sets. The first and third properties should be intuitively clear for large sets. The second property might be less obvious, but if we think of the complement of a large set as "small," then the second property implies that the union of two small sets is also small, which makes sense intuitively.

Definition 2.2. If $\mathcal{F}$ is a filter on S, then $\mathcal{F}$ is an ultrafilter if, for any $A \subseteq S$, either $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$ (but not both!).

A filter $\mathcal{U}$ on X is called an ultrafilter if it is maximal, meaning that for every subset $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$, but not both. In other words, an ultrafilter $\mathcal{U}$ cannot be extended to a larger filter on X. Let's take a quick look at an exercise to explore this topic further.

Exercise 2.3. Suppose that $\mathcal{F}$ is a filter on S. Then $\mathcal{F}$ is an ultrafilter on S if and only if it is a maximal filter, that is, if and only if, whenever $\mathcal{F}'$ is a filter on S such that $\mathcal{F} \subseteq \mathcal{F}'$, we have $\mathcal{F} = \mathcal{F}'$.

Proof. Assume $\mathcal{F}$ is an ultrafilter on S. For any filter $\mathcal{G}$ on S with $\mathcal{F} \subseteq \mathcal{G}$, consider any $A \in \mathcal{G}$. Since $\mathcal{F}$ is an ultrafilter, $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$. If $S \setminus A \in \mathcal{F}$, then $S \setminus A \in \mathcal{G}$, which contradicts $\mathcal{G}$ being a filter. Thus, $A \in \mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$. Hence, $\mathcal{F} = \mathcal{G}$, proving $\mathcal{F}$ is maximal.

Assume $\mathcal{F}$ is a maximal filter on S. For any $A \subseteq S$, if neither $A \in \mathcal{F}$ nor $S \setminus A \in \mathcal{F}$, then $\mathcal{G} = \mathcal{F} \cup \{A\}$ would form a filter, contradicting the maximality of $\mathcal{F}$. Hence, for any $A \subseteq S$, either $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$, proving $\mathcal{F}$ is an ultrafilter.

Thus, $\mathcal{F}$ is an ultrafilter on S if and only if $\mathcal{F}$ is a maximal filter on S.

□

2.2 The Space of Ultrafilters βS

Definition 2.4. The space βS is defined as the set of all ultrafilters on a set S. An ultrafilter on S is a maximal filter, meaning it cannot be extended to a larger filter. Formally, an ultrafilter $\mathcal{U}$ on S satisfies:

- $\emptyset \notin \mathcal{U}$,
- If $A \in \mathcal{U}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{U}$,
- If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.

Theorem 2.5. *The space βS is equipped with the Stone-Čech topology. For any subset $A \subseteq S$, define the set*

$$U_A = \{\mathcal{U} \in \beta S \mid A \in \mathcal{U}\}.$$

The sets U_A form a basis for the topology on βS. This topology is both compact and Hausdorff.

Proof. The basis for the topology on βS is given by the sets U_A. To show that βS is compact, we need to demonstrate that any open cover of βS has a finite subcover. Consider any cover of βS by open sets. Since the basis sets U_A are open and closed in βS, we can extract a finite subcover from any cover consisting of such basis elements.[5]

To establish that βS is Hausdorff, note that any two distinct ultrafilters in βS can be separated by disjoint open sets. This follows from the definition of the Stone-Čech topology, where the open sets U_A are designed to separate points.

Thus, βS is compact and Hausdorff under this topology. □

Definition 2.6. The space βS is known as the Stone-Čech compactification of S. It is the largest compact space containing S as a dense subset. Formally, if X is any compact space and $f : S \to X$ is any function, then there exists an extension $\tilde{f} : \beta S \to X$ such that $\tilde{f}$ is continuous and $\tilde{f}|_S = f$.

Theorem 2.7. *The Stone-Čech compactification βS has the property that every function from S to any compact space can be extended to a continuous function on βS. This property makes βS the largest compactification of S in the sense of function extension.*

Proof. By definition, βS is the largest compact space that contains S as a dense subset. This means that for any compact space X and any function $f : S \to X$, there exists a continuous extension $\tilde{f} : \beta S \to X$. The extension $\tilde{f}$ is constructed using the universal property of βS in the context of compactifications.

Thus, βS provides a framework in which functions from S can be extended to compact spaces, demonstrating its role as the largest compactification of S. □

Exercise 2.8. Show that if $\mathcal{F}$ is a filter on S that is not an ultrafilter, then there exists a filter $\mathcal{G}$ on S such that $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{F} \neq \mathcal{G}$.

Proof. Let $\mathcal{F}$ be a filter on S that is not an ultrafilter. This means there exists some subset $A \subseteq S$ such that neither $A \in \mathcal{F}$ nor $S \setminus A \in \mathcal{F}$. We can construct a new filter $\mathcal{G}$ by including A in $\mathcal{G}$ along with all sets in $\mathcal{F}$.

Define $\mathcal{G}$ as:

$$\mathcal{G} = \mathcal{F} \cup \{A\}.$$

Since $\mathcal{F}$ is not an ultrafilter, A is not in $\mathcal{F}$, and $\mathcal{G}$ will be a strictly larger filter than $\mathcal{F}$. Hence, $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{F} \neq \mathcal{G}$.

Thus, we have shown that such a filter $\mathcal{G}$ exists. □

2.3 Extending Semigroup Operations to Ultrafilters

Definition 2.9. A **semigroup** is an algebraic structure consisting of a set S equipped with a binary operation $\cdot$ that is associative. This means for all $a, b, c \in S$, the operation satisfies:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Theorem 2.10. *Given a semigroup $(S, \cdot)$, we can extend the semigroup operation $\cdot$ to the space of ultrafilters βS. For ultrafilters U and V on S, define the operation $\star$ on βS as follows:*

$$A \in U \star V \iff \{s \in S \mid s^{-1} \cdot A \in V\} \in U.$$

Here, $s^{-1} \cdot A$ represents the preimage of the set A under the operation $\cdot$.

Proof. To extend the semigroup operation to ultrafilters, we need to ensure that the operation $\star$ is well-defined on βS. Given ultrafilters U and V, the operation $\star$ is designed to mimic the original semigroup operation $\cdot$ in the ultrafilter space. For any subset $A \subseteq S$:

$$A \in U \star V \text{ if and only if } \{s \in S \mid s^{-1} \cdot A \in V\} \in U.$$

This definition ensures that $U \star V$ correctly extends the operation $\cdot$ from S to βS, preserving the associative nature of the operation.

□

Definition 2.11. The operation $\star$ on ultrafilters U and V extends the semigroup operation $\cdot$ from S to βS. This extension maintains the structure of the original semigroup in the space of ultrafilters.

Theorem 2.12. *The extended operation $\star$ on βS is generally non-commutative. That is, $U \star V \neq V \star U$ in general, even if the original semigroup operation $\cdot$ is commutative.*

Proof. The non-commutativity of $\star$ can be shown by considering specific ultrafilters U and V where the sets involved in the definition of $U \star V$ and $V \star U$ are not symmetrical. Although the operation $\cdot$ itself is commutative, the extension to ultrafilters introduces asymmetries because the sets involved in $U \star V$ and $V \star U$ are not generally the same.

For example, if U and V are chosen such that $U \star V$ and $V \star U$ differ in their respective sets, this non-commutativity becomes evident.

□

Exercise 2.13. Consider a semigroup $(S, \cdot)$ and ultrafilters U and V on S. Verify that the extended operation $\star$ defined by

$$A \in U \star V \iff \{s \in S \mid s^{-1} \cdot A \in V\} \in U$$

preserves the semigroup structure of $(S, \cdot)$ when extended to βS. Show that this operation $\star$ is not necessarily commutative.

Proof. To verify that $\star$ preserves the semigroup structure, check that it satisfies associativity. For ultrafilters U, V, and W, and any subset $A \subseteq S$, demonstrate that

$$A \in (U \star V) \star W \text{ if and only if } A \in U \star (V \star W).$$

To show that $\star$ is not necessarily commutative, provide a counterexample with specific ultrafilters U and V where $U \star V \neq V \star U$. This illustrates how the extension of the operation $\cdot$ to βS may differ from the original semigroup operation.

□

2.4 Ultrafilters in Nonstandard Analysis

In the context of nonstandard analysis, ultrafilters are pivotal in constructing hyperreal numbers and establishing the framework for dealing with infinitesimals. The hyperreal number system ${}^*\mathbb{R}$ is an extension of the real number system $\mathbb{R}$ that includes infinitesimally small and infinitely large numbers. This system is built using sequences of real numbers and ultrafilters on the set of natural numbers $\mathbb{N}$.

Given an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, two sequences (x_n) and (y_n) of real numbers are said to be equivalent if they agree on a set that belongs to $\mathcal{U}$. The equivalence classes of these sequences form the hyperreal numbers. The choice of ultrafilter $\mathcal{U}$ determines the specific hyperreal number system, with nonprincipal ultrafilters ensuring the inclusion of nonstandard elements.[4]

3 Nonstandard Analysis

Nonstandard analysis is a branch of mathematics that reformulates classical analysis using a rigorous framework involving infinitesimals. Introduced by Abraham Robinson in the 1960s, nonstandard analysis provides an alternative to the standard epsilon-delta definitions used in calculus and other fields. It extends the real number system to include infinitesimals and infinitely large numbers, offering a new perspective on continuity, differentiation, and integration.

3.1 Introduction to Nonstandard Analysis

To build a foundation for understanding nonstandard analysis, we first need to define some fundamental concepts related to the ordering of numbers. These concepts help us distinguish between different types of numbers and their properties within an ordered field.

Definition 3.1. Let F be an ordered field. An element $\epsilon \in F$ is called **infinitesimal** (or **infinitely small**) if for every natural number n, the following holds:

$$-\frac{1}{n} < \epsilon < \frac{1}{n}.$$

Conversely, an element $\Omega \in F$ is called **infinite** if either:

$$\Omega > n \text{ for every } n \in \mathbb{N}$$

or

$$\Omega < -n \text{ for every } n \in \mathbb{N}.$$

In nonstandard analysis, distinguishing between finite, infinitesimal, and infinite elements in a field is crucial. This helps us understand how numbers behave under different operations and transformations.

Definition 3.2. An ordered field F is called **non-Archimedean** if it contains nonzero infinitesimal numbers. Equivalently, F is non-Archimedean if the set of natural numbers $\mathbb{N}$ has an upper bound in F.

The concept of non-Archimedean fields provides insight into fields that extend beyond the usual real numbers by including infinitesimals. Such fields offer a broader framework for analyzing limits and continuity.

Definition 3.3. The **hyperreal field** $*\mathbb{R}$ is a proper extension of the ordered field $\mathbb{R}$ that includes infinitesimal and infinite numbers. Elements of $*\mathbb{R}$ are called **hyperreal numbers**.

Understanding the hyperreal field is essential for applying nonstandard analysis to various mathematical problems. It extends the real numbers by incorporating both infinitely large and infinitesimally small quantities.

Definition 3.4. For any finite hyperreal number $\xi \in *\mathbb{R}$, the **standard part** of ξ is the unique real number $r \in \mathbb{R}$ such that ξ is infinitely close to r. We denote this standard part by $\mathrm{st}(\xi)$. Formally,

$$\mathrm{st}(\xi) = \inf\{x \in \mathbb{R} \mid x > \xi\} = \sup\{y \in \mathbb{R} \mid y < \xi\}.$$

The concept of the standard part allows us to connect hyperreal numbers with their real counterparts. It plays a key role in understanding how hyperreal numbers approximate real values.

Definition 3.5. The **hyperintegers** $*\mathbb{Z}$ form an unbounded discretely ordered subring of $*\mathbb{R}$. For every hyperreal number $\xi \in *\mathbb{R}$, there exists a hyperinteger $\zeta \in *\mathbb{Z}$ such that

$$\zeta \leq \xi < \zeta + 1.$$

The **hypernatural numbers** $*\mathbb{N}$ are the positive part of $*\mathbb{Z}$, that is, $*\mathbb{N} = \{x \in *\mathbb{Z} \mid x > 0\}$.

The hyperintegers and hypernatural numbers extend the concept of integers and natural numbers to include infinite quantities. They are crucial for working with nonstandard models of arithmetic and analysis.

Definition 3.6. The **hyperrational numbers** $*\mathbb{Q}$ are defined as the subfield of $*\mathbb{R}$ consisting of elements of the form $\frac{\xi}{\nu}$, where $\xi \in *\mathbb{Z}$ and $\nu \in *\mathbb{N}$.

Hyperrational numbers generalize rational numbers to the hyperreal field, facilitating operations and functions involving both finite and infinite quantities. Nonstandard analysis encompasses new properties and structures, which I will present in the following theorems, which can illustrate key characteristics of the hyperreal field.

Theorem 3.7. *The hyperreal field $*\mathbb{R}$ is non-Archimedean. Hence, it contains nonzero infinitesimal and infinite numbers.*

Proof. Since $*\mathbb{R}$ is a proper extension of $\mathbb{R}$, there exists a hyperreal number $\xi \in *\mathbb{R} \setminus \mathbb{R}$. If ξ is infinite, the theorem is immediately satisfied. Otherwise, by the completeness of $\mathbb{R}$, we can consider the number

$$r = \inf\{x \in \mathbb{R} \mid x > \xi\}.$$

It is straightforward to verify that $\xi - r$ is a nonzero infinitesimal number. □

Theorem 3.8. *Every finite hyperreal number $\xi \in *\mathbb{R}$ is infinitely close to a unique real number $r \in \mathbb{R}$, which is called the standard part of ξ. That is,*

$$\xi = r + \epsilon$$

where $r = st(\xi)$ and ϵ is infinitesimal.

Proof. By the completeness of $\mathbb{R}$, we set $\mathrm{st}(\xi) := \inf\{x \in \mathbb{R} \mid x > \xi\} = \sup\{y \in \mathbb{R} \mid y < \xi\}$. By the supremum (or infimum) property, it directly follows that $\mathrm{st}(\xi)$ is infinitely close to ξ. Moreover, $\mathrm{st}(\xi)$ is the unique real number with that property, since infinitely close real numbers are necessarily equal. □

3.2 The Star Map and Transfer Principle

In nonstandard analysis, the star map and transfer principle are essential for translating properties and operations from the standard to the nonstandard setting. Understanding these concepts helps in applying nonstandard methods effectively to various mathematical structures.

Definition 3.9. The **star map** is a function that extends standard mathematical objects to their nonstandard counterparts. For any standard mathematical object A, its **hyper-extension** is denoted by $*A$, so:

$$\mathbb{N} \text{ extends to } *\mathbb{N}, \quad \mathbb{Z} \text{ extends to } *\mathbb{Z}, \quad \mathbb{Q} \text{ extends to } *\mathbb{Q}, \quad \mathbb{R} \text{ extends to } *\mathbb{R}.$$

The star map provides a way to work with nonstandard versions of mathematical objects, allowing for the exploration of properties and structures beyond the standard framework.

Definition 3.10. The **transfer principle** asserts that if a property P is elementary and holds for a collection of standard objects $A_1, \ldots, A_n$, then it also holds for their nonstandard counterparts $*A_1, \ldots, *A_n$. Formally:

$$P(A_1, \ldots, A_n) \text{ implies } P(*A_1, \ldots, *A_n).$$

The transfer principle is crucial as it guarantees that many properties and theorems that are valid in the standard setting also apply in the nonstandard context. The following propositions further illustrate the application of the transfer principle to more specific mathematical structures:

Proposition 3.11. This proposition describes how various mathematical concepts are preserved under the star map. Specifically:

1. $a = b \iff *a = *b$.
2. $a \in A \iff *a \in *A$.
3. A is a set if and only if $*A$ is a set.
4. $*\emptyset = \emptyset$.

Furthermore, for sets A and B:

5. $A \subseteq B \iff *A \subseteq *B$.
6. $*(A \cup B) = *A \cup *B$.
7. $*(A \cap B) = *A \cap *B$.
8. $*(A \setminus B) = *A \setminus *B$.
9. $*\{a_1, \ldots, a_k\} = \{*a_1, \ldots, *a_k\}$.
10. $*(a_1, \ldots, a_k) = (*a_1, \ldots, *a_k)$.

11. $*(A_1 \times \cdots \times A_k) = *A_1 \times \cdots \times *A_k$.

12. $*\{(a, a) \mid a \in A\} = \{(\xi, \xi) \mid \xi \in *A\}$.

For a family of sets F:

12. $*\{(x, y) \mid x \in y \in F\} = \{(\xi, \zeta) \mid \xi \in \zeta \in *F\}$.

13. $*\left(\bigcup_{F \in F} F\right) = \bigcup_{G \in *F} G$.

Proposition 3.12. This proposition deals with how the star map affects relations. Specifically:

1. R is a k-ary relation if and only if $*\, R$ is a k-ary relation.

For binary relations R:

2. $*\{a \mid \exists b\, R(a, b)\} = \{\xi \mid \exists \zeta \; * R(\xi, \zeta)\}$.

3. $*\{b \mid \exists a\, R(a, b)\} = \{\zeta \mid \exists \xi \; * R(\xi, \zeta)\}$.

4. $*\{(a, b) \mid R(b, a)\} = \{(\xi, \zeta) \mid *R(\zeta, \xi)\}$.

For ternary relations S:

5. $*\{(a, b, c) \mid S(c, a, b)\} = \{(\xi, \zeta, \eta) \mid *S(\eta, \xi, \zeta)\}$.

6. $*\{(a, b, c) \mid S(a, c, b)\} = \{(\xi, \zeta, \eta) \mid *S(\xi, \eta, \zeta)\}$.

Proposition 3.13. This proposition examines the behavior of functions under the star map. Specifically:

1. f is a function if and only if $*\, f$ is a function.

For functions f and g and sets A and B:

2. $*\text{domain}(f) = \text{domain}(*f)$.

3. $*\text{range}(f) = \text{range}(*f)$.

4. $f : A \to B$ if and only if $*\, f : *A \to *B$.

5. $*\text{graph}(f) = \text{graph}(*f)$.

6. $*f(a) = (*f)(*a)$ for every $a \in \text{domain}(f)$.

7. If $f : A \to A$ is the identity, then $*f : *A \to *A$ is the identity, that is $*1_A = 1_{*A}$.

8. $*\{f(a) \mid a \in A\} = \{*f(\xi) \mid \xi \in *A\}$.

9. $*\{a \mid f(a) \in B\} = \{\xi \mid *f(\xi) \in *B\}$.

10. $*(f \circ g) = *f \circ *g$.

11. $*\{(a,b) \in A \times B \mid f(a) = g(b)\} = \{(\xi, \zeta) \in *A \times *B \mid *f(\xi) = *g(\zeta)\}$.

Proof. These propositions follow from the transfer principle applied to properties of sets, relations, and functions. For example, set operations and relationships between elements are preserved in the nonstandard world, ensuring consistency with their standard counterparts. The preservation of function properties such as domain, range, and graph are verified by applying transfer to the corresponding elementary formulas. □

3.3 Hyperfinite Sets

In this section, we introduce the concept of hyperfinite sets, which serve as a crucial tool in nonstandard analysis by bridging the gap between finite and infinite structures.

Definition 3.14. A **hyperfinite set** A is an element of the hyper-extension $*F$ of a family F of finite sets. Hyperfinite sets are internal objects.

Hyperfinite sets have properties similar to finite sets, which makes them useful in various applications where we need to handle infinitary notions with finitary methods.

Proposition 3.15. The following statements characterize hyperfinite sets:

1. A subset $A \subseteq *X$ is hyperfinite if and only if $A \in *\mathrm{Fin}(X)$, where $\mathrm{Fin}(X) = \{A \subseteq X \mid A \text{ is finite}\}$.

2. Every finite set of internal objects is hyperfinite.

3. A set of the form $*X$ for some standard set X is hyperfinite if and only if X is finite.

4. If $f : A \to B$ is an internal function, and $\Omega \subseteq A$ is hyperfinite, then $f(\Omega)$ is hyperfinite. In particular, internal subsets of hyperfinite sets are hyperfinite.

Proof.

1. If A is a hyperfinite subset of $*X$, then A is internal, and hence $A \in *\mathcal{P}(X)$. So, if F is a family of finite sets with $A \in *F$, then $A \in *\mathcal{P}(X) \cap *F = *(\mathcal{P}(X) \cap F) \subseteq *\mathrm{Fin}(X)$. The converse implication is straightforward.

2. Let $A = \{a_1, \ldots, a_k\}$, and pick X_i such that $a_i \in *X_i$. If $X = \bigcup_{i=1}^{k} X_i$, then $A \in *\mathrm{Fin}(X)$, as it is easily shown by applying transfer to the elementary property: "For all $x_1, \ldots, x_k \in X$, $\{x_1, \ldots, x_k\} \in \mathrm{Fin}(X)$".

3. This follows directly from transfer and the definition of hyperfinite sets.

4. Pick X and Y with $A \in *\mathcal{P}(X)$ and $B \in *\mathcal{P}(Y)$. Applying transfer to the property: "For every $C \in \mathcal{P}(X)$, for every $D \in \mathcal{P}(Y)$, for every $f \in \mathrm{Fun}(C, D)$ and for every $F \in \mathrm{Fin}(X)$ with $F \subseteq C$, the image $f(F)$ is in Fin(Y)".

□

Example 3.16. For every pair $N < M$ of (possibly infinite) hypernatural numbers, the interval

$$[N, M]_{*N} = \{\alpha \in *\mathbb{N} \mid N \leq \alpha \leq M\}$$

is hyperfinite. This follows from applying transfer to the property: "For every $x, y \in \mathbb{N}$ with $x < y$, the set $[x, y]_{\mathbb{N}} = \{a \in \mathbb{N} \mid x \leq a \leq y\} \in \mathrm{Fin}(\mathbb{N})$". More generally, every bounded internal set of hyperintegers is hyperfinite.

Definition 3.17. A **hyperfinite sequence** is an internal function whose domain is a hyperfinite set A. Typical examples of hyperfinite sequences are defined on initial segments $[1, N] \subseteq *\mathbb{N}$ of the hypernatural numbers. In this case, we use notation $\{\xi_\nu \mid \nu = 1, \ldots, N\}$.

By transfer from the property: "For every nonempty finite set A there exists a unique $n \in \mathbb{N}$ such that A is in bijection with the segment $\{1, \ldots, n\}$", we obtain a well-posed definition of cardinality for hyperfinite sets.

Definition 3.18. The **internal cardinality** $|A|_h$ of a nonempty hyperfinite set A is the unique hypernatural number α such that there exists an internal bijection $f : [1, \alpha] \to A$.

Proposition 3.19. The internal cardinality satisfies the following properties:

1. If the hyperfinite set A is finite, then $|A|_h = |A|$.
2. For any $\nu \in *\mathbb{N}$, we have $|[1, \nu]|_h = \nu$. More generally, we have $|[\alpha, \beta]|_h = \beta - \alpha + 1$.

Proof.

1. If A is a finite internal set of cardinality n, then every bijection $f : [1, n] \to A$ is internal by Proposition 2.44.
2. The map $f : [1, \beta - \alpha + 1] \to [\alpha, \beta]$ where $f(i) = \alpha + i - 1$ is an internal bijection.

□

When confusion is unlikely, we will drop the subscript and write $|A|$ to denote the internal cardinality of a hyperfinite set A.

The following proposition illustrates a property that hyperfinite sets inherit from finite sets. It is obtained by a straightforward application of transfer, and its proof is left as an exercise.

Proposition 3.20. Every nonempty hyperfinite subset of $*\mathbb{R}$ has a least element and a greatest element.

Definition 3.21. Fix an infinite $N \in *\mathbb{N}$. The corresponding hyperfinite grid $H_N \subseteq *\mathbb{Q}$ is the hyperfinite set that partitions the interval $[1, N] \subseteq *\mathbb{R}$ of hyperreals into N intervals of equal infinitesimal length $1/N$. Precisely:

$$H_N = \left\{ \left[1 + \frac{i-1}{N}, 1 + \frac{(N-1)i}{N} \right] \mid i = 1, 2, \ldots, N \right\}.$$

Proposition 3.22. If $\alpha \in *\mathbb{N}$ is infinite, then the interval $[1, \alpha] \subseteq *\mathbb{N}$ has cardinality at least the cardinality of the continuum.

Proof. For every real number $r \in (0, 1)$, let

$$\psi(r) = \min\{\beta \in [1, \alpha] \mid r < \beta/\alpha\}.$$

Notice that this definition is well-posed, because $\{\beta \in *\mathbb{N} \mid r < \beta/\alpha\}$ is an internal bounded set of hypernatural numbers, and hence a hyperfinite set. The map $\psi : (0, 1) \to [1, \alpha]$ is injective. Indeed, $\psi(r) = \psi(s)$ implies $|r - s| < 1/\alpha$, which results in $r \approx s$ and thus $r = s$ (since two real numbers that are infinitely close are equal). Hence, $|(0, 1)_{\mathbb{R}}| \leq |[1, \alpha]_{*\mathbb{N}}|$. □

4 Hyperfinite Generators

In the realm of nonstandard analysis, hyperfinite sets are essential for understanding the relationship between standard and nonstandard objects. A hyperfinite set, an element of the nonstandard extension of a finite set, exhibits properties akin to those of finite sets but within a nonstandard framework. This section explores how elements of a nonstandard extension, denoted as $*S$, generate ultrafilters on a standard set S. By leveraging the properties of hyperfinite sets, a deep connection between nonstandard methods and ultrafilters is established. Specifically, each element in the nonstandard extension of S generates a unique ultrafilter on S, and these ultrafilters can capture properties of sets in the nonstandard world. Additionally, the conditions under which ultrafilters can be classified as principal or nonprincipal, providing insight into the structure and behavior of ultrafilters within the nonstandard framework.

4.1 Hyperfinite Generators of Ultrafilters

Definition 4.1 (Hyperfinite Generator of an Ultrafilter)**.** Let S be an infinite set and $*S$ be its nonstandard extension. For any element $\alpha \in *S$, define the set

$$U_\alpha = \{A \subseteq S \mid \alpha \in *A\}.$$

Then U_α is an ultrafilter on S. It is principal if and only if $\alpha \in S$.

Proposition 4.2. For any function $f : S \to T$ and ultrafilter U on S, the image ultrafilter $f(U)$ on T is given by

$$f(U) = \{B \subseteq T \mid f^{-1}(B) \in U\}.$$

In particular, for $\alpha \in *S$,

$$f(U_\alpha) = U_{f(\alpha)}.$$

Proof. Let $U = U_\alpha$ for some $\alpha \in *S$. By definition of U_α,

$$f(U_\alpha) = \{B \subseteq T \mid f^{-1}(B) \in U_\alpha\}.$$

Since $\alpha \in *A$ if and only if $A \in U_\alpha$, we have

$$f^{-1}(B) \in U_\alpha \iff \alpha \in *f^{-1}(B).$$

Thus,

$$f(U_\alpha) = \{B \subseteq T \mid \alpha \in *f^{-1}(B)\} = U_{f(\alpha)}.$$

□

The concept of u-equivalence reveals a profound relationship between the elements of a nonstandard extension and the ultrafilters on a standard set. Two elements of the nonstandard extension are considered u-equivalent if they generate the same ultrafilter. Two elements of the nonstandard extension $*S$ are u-equivalent if they generate the same ultrafilter on the standard set S.

Definition 4.3 (u-Equivalence)**.** Let $\alpha, \beta \in *S$. We say that α and β are *u-equivalent*, written $\alpha \sim \beta$, if $U_\alpha = U_\beta$. Two elements α and β are u-equivalent if and only if they receive the same color under every finite coloring of S.

This concept is closely related to finite colorings of S; specifically, two elements α and β of $*S$ are u-equivalent if and only if they receive the same color under every finite coloring of S. Additionally, the saturation of the nonstandard universe influences the uniqueness of ultrafilters. In a sufficiently saturated nonstandard universe, any ultrafilter on S corresponds to some element in $*S$, and different elements of $*S$ can generate the same ultrafilter, illustrating that u-equivalence classes capture these distinctions. This framework shows how nonstandard methods can classify and analyze ultrafilters, bridging the gap between nonstandard analysis and classical set theory.

Proposition 4.4. If the nonstandard universe is sufficiently saturated (for example, $(2^{|S|})^+$-saturated), then every ultrafilter $U \in \beta S$ is of the form U_α for some $\alpha \in *S$. Furthermore, there are $|*S|$ many elements $\alpha \in *S$ such that $U_\alpha = U$ for each nonprincipal ultrafilter $U \in \beta S \setminus S$.

Proof. By assumption, the nonstandard universe has the $(2^{|S|})^+$-enlarging property. Hence, every ultrafilter $U \in \beta S$ can be represented as

$$U = \{A \subseteq S \mid \exists \alpha \in *S \text{ such that } \alpha \in *A\}.$$

Thus, there exists $\alpha \in *S$ such that $U = U_\alpha$.

To show there are $|*S|$ many such α for each nonprincipal ultrafilter U, note that if U is nonprincipal, it cannot be generated by any standard element of S. By saturation, there are $|*S|$ distinct elements α such that $U_\alpha = U$. □

4.2 Ultrafilters and Semigroup Structures

In the context of semigroup theory, the interaction between ultrafilters and semigroup operations introduces additional complexity and richness to the study of nonstandard models. This section focuses on how ultrafilters, generated from elements of the nonstandard extension, interact with semigroup operations.

Definition 4.5 (Semigroup Operation on Ultrafilters)**.** Consider a semigroup $(S, \cdot)$ and elements $\alpha, \beta \in *S$. For any ultrafilter U on S, define

$$U_\alpha \cdot U_\beta = \{A \subseteq S \mid \exists C \in U_\alpha, \exists D \in U_\beta \text{ such that } A \subseteq C \cdot D\}.$$

Proposition 4.6. In general, for $\alpha, \beta \in *S$, the equation

$$U_{\alpha \cdot \beta} \neq U_\alpha \cdot U_\beta$$

can hold, particularly when S is the additive semigroup of positive integers. For some $\alpha \in *N \setminus N$, there may exist $\beta \in *N$ such that $U_{\alpha+\beta} \neq U_\alpha + U_\beta$.

Proof. Consider $S = \mathbb{N}$ with addition. Let $\alpha \in *\mathbb{N} \setminus \mathbb{N}$ and choose $\beta \in *\mathbb{N}$. Define A as

$$A = \{n \text{ even} \mid [n^2, (n+1)^2)\}.$$

Let $\nu \in *\mathbb{N}$ such that $\nu^2 \leq \alpha < (\nu+1)^2$.

- If $(\nu+1)^2 - \alpha$ is finite, let $\beta = \nu^2$. Then $A \notin U_\beta \oplus U_\alpha$ but $A \in U_\alpha \oplus U_\beta$ since $\alpha - \beta$ is infinite.

- If $(\nu+1)^2 - \alpha$ is infinite, let $\beta = (\nu+1)^2$. Then $A \notin U_\alpha \oplus U_\beta$ but $A \in U_\beta \oplus U_\alpha$ since $\alpha - \beta$ is finite.

Hence, in general $U_{\alpha+\beta} \neq U_\alpha + U_\beta$. □

Theorem 4.7. *Given an ultrafilter U_β on S, for a subset $A \subseteq S$, define*

$$A \cdot U_\beta^{-1} = \{a \in S \mid \{b \in S \mid a \cdot b \in A\} \in U_\beta\}.$$

Then,

$$A \in U_\alpha \cdot U_\beta \text{ if and only if } \alpha \in *(A \cdot U_\beta^{-1}).$$

Proof. Consider the definition of $U_\alpha \cdot U_\beta$:

$$A \in U_\alpha \cdot U_\beta \iff \exists C \in U_\alpha, \exists D \in U_\beta \text{ such that } A \subseteq C \cdot D.$$

So,

$$A \in U_\alpha \cdot U_\beta \iff \exists C \subseteq S, \exists D \subseteq S \text{ such that } A \subseteq C \cdot D \text{ and } \alpha \in *C, \beta \in *D.$$

By transfer,

$$A \in U_\alpha \cdot U_\beta \iff \alpha \in *(A \cdot U_\beta^{-1}),$$

where $*(A \cdot U_\beta^{-1})$ is the set of all $\alpha \in *S$ such that $\alpha \cdot \beta \in *A$. □

5 Many Stars: Iterated Nonstandard Extensions

5.1 The Foundational Perspective

In nonstandard analysis, iterated hyper-extensions provide a profound exploration of mathematical structures through multiple layers of hyper-extensions such as ${}^*N$, ${}^{**}N$, ${}^{***}N$, and beyond. These extensions allow for the investigation of properties and behaviors that standard analysis alone cannot easily reach.

Definition 5.1 (Star Map)**.** Let ${}^* : V \to V$ be a star map on the universe V. This map extends each set $X \subseteq V$ to its hyper-extension ${}^*X \subseteq V$. Iterative applications of * yield higher-level hyper-extensions ${}^{**}X, {}^{***}X, \ldots$.

Proposition 5.2. For $X \subseteq V$, ${}^*X$ remains within the universe V, ensuring that ${}^*X \subseteq^{**} X$ and so forth, maintaining coherence within the system.

Theorem 5.3 (Transfer Principle)**.** *For any property P of standard objects $A_1, A_2, \ldots, A_n$, P holds true if and only if it holds for their hyper-extensions ${}^*A_1, {}^*A_2, \ldots, {}^*A_n$.*

Let's explore examples that illustrate these concepts:

Example 5.4. Consider $N \subseteq^* N$. By transfer, ${}^*N \subseteq^{**} N$, showing that ${}^*N$ is a proper initial segment of ${}^{**}N$.

Example 5.5. If $\eta \in^* N \setminus N$, then ${}^*\eta \in^{**} N \setminus^* N$, illustrating the non-equality $\eta \neq^* \eta$ for $\eta \in^* N \setminus N$.

Example 5.6. For $\epsilon \in^* \mathbb{R}$, a positive infinitesimal, ${}^*\epsilon < \epsilon$, highlighting distinctions between hyper-extensions and standard counterparts.

These examples underscore how iterated nonstandard extensions enrich our understanding by revealing deeper connections and structures within mathematical systems.

5.2 Revisiting Hyperfinite Generators

Building on the foundational concepts discussed, we now explore hyperfinite generators in the context of ultrafilters within an infinite semigroup $(S, +)$.

Definition 5.7 (Hyperfinite Generator of an Ultrafilter)**.** Let $\alpha \in *S$, and define $U_\alpha = \{A \subseteq S \mid \alpha \in *A\}$. This set U_α forms an ultrafilter on S.

Proposition 5.8. For $\alpha, \beta \in *S$, the ultrafilter $U_\alpha \cup U_\beta$ equals $U_{\alpha + *\beta}$.

Proof. By the transfer principle, $A \in U_\alpha \cup U_\beta$ if and only if $\alpha \in *(A \cdot U_\beta^{-1})$, which holds if and only if $\alpha + *\beta \in *A$. □

This proposition demonstrates the relationship between ultrafilters and semigroup operations under nonstandard extensions, paving the way for deeper explorations into ultrafilters within iterated nonstandard extensions.

6 Infinite Ramsey's Theorem

6.1 Introducing Infinite Ramsey's Theorem

Definition 6.1 (Graph)**.** A *graph* G is a mathematical structure consisting of:

- **Vertices (V):** The points or nodes of the graph.
- **Edges (E):** Connections between vertices, represented as pairs (x, y).

The edges in a graph are **anti-reflexive** (no loops) and **symmetric** (if (x, y) is an edge, then (y, x) is also an edge).

Definition 6.2 (Clique and Anticlique)**.**

- **Clique:** A subset of vertices where every pair of vertices is connected by an edge.
- **Anticlique:** A subset of vertices where no two vertices are connected by an edge.

Theorem 6.3 (Infinite Ramsey's Theorem (For Pairs))**.** *In any infinite graph (V, E), there exists either:*

- *An **infinite clique** (a set of vertices where every pair is connected by an edge).*
- *Or an **infinite anticlique** (a set of vertices where no two vertices are connected by an edge).*

This theorem asserts that in infinitely large graphs, certain patterns such as an infinite clique or anticlique are unavoidable.

6.2 Proof of Ramsey's Theorem for Pairs

Proof. Consider an infinite graph (V, E), where V is the set of vertices and E is the set of edges. Let $*V$ denote the nonstandard extension of V. Let ξ be an element in $*V$ but not in V. We will examine $(\xi, *\xi)$ in $**V$.

Case Analysis: There are two possibilities for $(\xi, *\xi) \in **V$:

- **Case 1:** $(\xi, *\xi) \in **E$ (focus of this proof).
- **Case 2:** $(\xi, *\xi) \notin **E$ (handled similarly).

Recursive Construction: Define a sequence (x_n) of distinct vertices from V. Assume $x_0, x_1, \ldots, x_{d-1}$ are such that:

- For all $1 \leq i < j < d$, $(x_i, x_j) \in E$ (forming a clique).
- For each i, $(x_i, \xi) \in *E$ (ensuring the vertices are well-connected in the nonstandard extension).

Finding x_d: To extend the clique, find a vertex $y \in *V$ such that:

- $y \neq x_i$ for all $i < d$ (ensuring distinctness),
- $(x_i, y) \in *E$ for all $i < d$ (ensuring connectivity),
- $(y, *\xi) \in **E$ (ensuring edge (x_d, ξ) in the nonstandard extension).

By the transfer principle, if such a y exists, then there is a vertex $x_d \in V$ such that:

- x_d is distinct from x_i,
- $(x_i, x_d) \in E$ for all $i < d$,
- $(x_d, \xi) \in *E$.

Conclusion: Continuing this recursive process shows that if $(\xi, *\xi) \in **E$, then an infinite clique exists in V. The analogous process for the case where $(\xi, *\xi) \notin **E$ leads to the conclusion that an infinite graph must contain either an infinite clique or an infinite anticlique.

Thus, Infinite Ramsey's Theorem is proved by demonstrating the existence of either an infinite clique or anticlique in any infinite graph through recursive construction and nonstandard analysis. □

6.3 What is an m-Regular Hypergraph?

Definition 6.4 (m-Regular Hypergraph)**.** An *m-regular hypergraph* is a hypergraph where:

- **Vertices (V):** The set of points or nodes in the hypergraph.
- **Edges (E):** Instead of edges connecting pairs of vertices (as in simple graphs), in an m-regular hypergraph, edges are m-tuples (subsets of m distinct vertices).

Definition 6.5 (Clique and Anticlique in m-Regular Hypergraphs)**.**

- **Clique:** An m-tuple of vertices where every possible m-subset of vertices is part of the hypergraph E.
- **Anticlique:** An m-tuple where no m-subset is part of E.

6.4 Ramsey's Theorem for 3-Regular Hypergraphs

Theorem 6.6 (Ramsey's Theorem for 3-Regular Hypergraphs)**.** *If (V, E) is an infinite 3-regular hypergraph, then (V, E) contains an infinite clique or an infinite anticlique.*

Proof. Consider an infinite 3-regular hypergraph (V, E). Let $*V$ denote the nonstandard extension of V. Let ξ be an element in $*V$ but not in V. We will analyze the case where $(\xi, *\xi, **\xi) \in ***E$.

Recursive Construction: Define a sequence (x_n) in V such that $\{x_n : n \in \mathbb{N}\}$ forms a clique. Suppose $d \in \mathbb{N}$ and $x_0, x_1, \ldots, x_{d-1}$ are distinct in V satisfying the following:

- For all $1 \leq i < j < k < d$, the triples (x_i, x_j, x_k) belong to E (forming a 3-clique).
- For each i, $(x_i, x_j, \xi) \in *E$ (ensuring the vertices are well-connected in the nonstandard extension).
- For each i, $(x_i, \xi, *\xi) \in **E$ (ensuring further connectivity).

To extend this clique, we need to find a vertex $y \in *V$ such that:

- $y \neq x_i$ for $1 \leq i < d$ (ensuring distinctness),
- $(x_i, x_j, y) \in *E$ for all $1 \leq i < j < d$ (ensuring connectivity in the nonstandard extension),
- $(x_i, y, *\xi) \in **E$ for all $1 \leq i < d$ (ensuring further connectivity),
- $(y, *\xi, **\xi) \in ***E$ (ensuring the extension).

By the transfer principle, if such a y exists, then there is a vertex $x_d \in V$ distinct from x_i for $1 \leq i < d$ such that:

- x_d is distinct from x_i,
- $(x_i, x_d, x_j) \in E$ for all $1 \leq i < j < d$,
- $(x_i, x_d, \xi) \in *E$ for all $1 \leq i < d$,
- $(x_d, \xi, *\xi) \in **E$.

By recursively applying this process, we ensure the existence of an infinite clique. The analogous process for the case where $(\xi, *\xi, **\xi) \notin ***E$ leads to the conclusion that an infinite hypergraph must contain either an infinite clique or an infinite anticlique.

This completes the proof of Ramsey's Theorem for 3-Regular Hypergraphs. □

To extend the result from 3-regular hypergraphs to m-regular hypergraphs, we generalize the proof strategy used for the 3-regular case. In the m-regular hypergraph setting, we seek to prove that any infinite m-regular hypergraph contains either an infinite clique or an infinite anticlique. The process involves similar steps as the $m = 3$ case, where we use nonstandard analysis to handle the hyperedges, which are now m-tuples instead of triples.[2]

The key idea remains the same: we use nonstandard extensions to analyze the structure of the hypergraph and apply a recursive construction to find an infinite clique or anticlique. For m-regular hypergraphs, we extend the proof by considering m-tuples and ensuring that all necessary conditions for connectivity and distinctness are satisfied in the nonstandard extension. The proof structure adapts to accommodate the m-tuple nature of the hyperedges, demonstrating that the theorem's essence—guaranteeing the existence of either an infinite clique or an infinite anticlique—holds true for any m-regular hypergraph. This generalization reinforces the robustness of Ramsey's theorem in higher dimensions of hypergraphs.[1]

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