The Group Extension Problem

Ishan Joshi

Euler Circle

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What is the Group Extension Problem?

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The group extension problem for groups H and K asks us to find all groups G, such that $K \lhd G$, and $G/K \cong H$.

Intuitively, G can be factored into K and H.

Example

Let K be $\mathbb{Z}$, and $H = [0, 1) \bmod 1$. Then $\mathbb{R}$ is an extension G, of H by K. $K \times H$ is another extension.

Short Exact Sequence

To generalize the idea of a normal subgroup, we introduce a short exact sequence,

Definition 1

A sequence of groups:

$$1 \xrightarrow{f_1} K \xrightarrow{f_2} G \xrightarrow{f_3} H \xrightarrow{f_4} 1$$

Is called *short exact* if all f_i are homomorphisms and $\mathrm{im}(f_i) = \ker(f_{i+1})$.

G is an extension of H by K if G fits into the above short exact sequence.

Example

The sequence:

$$1 \xrightarrow{f_1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_3} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_4} 1$$

is short exact. To see this, we let $f_2(a) = (a, 0)$, and let $f_3(a, b) = b$, for all $a, b \in \mathbb{Z}/2\mathbb{Z}$.

Main Theorem

Theorem 1.1

There exists a bijection between $H^2_\theta(H, K)$ and the set of equivalence classes, E, of extensions which realize the data (H, K, θ).

As a result, the description of all congruence clases of extensions of H by K can be described as:

$$\bigcup_{\theta \in \mathrm{Hom}(H, \mathrm{Aut}(K))} H^2_\theta(H, K)$$

Sketch

- When $G/K \cong H$, it can be noted that H acts upon K.
- This homomorphism is determined by the congruence classes of the extensions of H by K.
- The congruence classes of extensions are able to be partitioned by the elements of $\mathrm{Hom}(H, \mathrm{Aut}(K))$
- For the fixed homomorphism θ, the congruence classes of extensions which induce θ is described by the second cohomology group $H^2_\theta(H, K)$.

Transversal

Definition 2

A transversal is a set containing one element from every coset of K. We denote this transversal T. Alternatively, it can also be thought of as a function $\phi : H \longrightarrow G$, where $\phi(H) = T$.

Lemma 3

Let T be a transversal of K in G. Then

$$G = \bigcup_{t \in T} tK$$

Example

$K = \mathbb{Z}$ and $H = [0, 1)$, we can see that a transversal of K in $\mathbb{R}$ would be the set $[0, 1)$.

Definition 4

An ordered triple (H, K, θ) is called *data* if K is abelian, H is a group, and $\theta : H \longrightarrow \mathrm{Aut}(K)$ is a homomorphism. If G is an extension of H by K, then G *realizes* the data if for every transversal $\phi : H \longrightarrow G$:

$$xa = \phi(x) + a - \phi(x).$$

For $a \in K$ and $x \in H$.

Factor Sets

Definition 5

A factor set for the group G is a function $f : H \times H \longrightarrow K$ such that for all $x, y \in H$ and some transversal ϕ,

$$\phi(x) + \phi(y) = f(x, y) + \phi(xy)$$

We call the set of factor sets $Z^2_\theta(H, K)$.

Example

Let $K = \mathbb{Z}$ and $H = (0, 1]$, with transversal $T = [0, 1)$, we define $\phi(h) = h$. This means ϕ is a homomorphism, so $f(x, y) = 0$, for all $x, y \in H$.

Construction of G_f

We take this time to provide a construction of a group realizing the data (H, K, θ) and factor set f. We consider the set $K \times H$, along with addition:

$$(k, x) + (l, y) = (k + xl + f(x, y), xy).$$

We denote this group as G_f, and the equivalence class of this group $[G_f]$.

Coboundaries

Definition 6

Given the data, (H, K, θ), a coboundary is a function $g : H \times H \longrightarrow K$, such that for all $x, y \in H$,

$$g(x, y) = xe(y) - e(xy) + e(x)$$

Where $e : H \longrightarrow K$ and $e(1) = 0$. The set of all coboundaries is an is denoted $B^2_\theta(H, K)$.

Second Cohomology Group

Definition 7

We define the second cohomology group to be $Z_\theta^2(H,K)/B_\theta^2(H,K)$. We denote this group as $H_\theta^2(H,K)$.

This is not the actual definition of the *ith* cohomology group, but it will suffice for now.

Main Theorem

Theorem 1.2

There exists a bijection between $H^2_\theta(H, K)$ and the set of equivalence classes, E, of extensions which realize the data (H, K, θ).

Proof sketch: We define a function $\rho : H^2_\theta(H, K) \longrightarrow E$, as $\rho(f + B^2_\theta(H, K)) = [G_f]$. We prove that ρ is bijective.

Diagram

This is a diagram depicting how we classify the equivalence classes of the extensions of H by K.

$$
\begin{tikzcd}
 & G \arrow[dl, "\theta_1"'] \arrow[d, "\theta_2"] \arrow[dr, "\theta_n"] & \\
{[G_{f_1}]} \arrow[d] & {[G_{f_2}]} \arrow[d] & {[G_{f_n}]} \arrow[d] \\
H^2_{\theta_1}(H, K) & H^2_{\theta_2}(H, K) & H^2_{\theta_n}(H, K)
\end{tikzcd}
$$

Thank you for your attention!

If you have any questions you can message me on discord.