

The Group Extension Problem

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What is the Group Extension Problem?

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The group extension problem for groups H and K asks us to find all groups G , such that $K \triangleleft G$, and $G/K \cong H$.

Intuitively, G can be factored into K and H .

Example

Let K be \mathbb{Z} , and $H = [0, 1) \bmod 1$. Then \mathbb{R} is an extension G , of H by K . $K \times H$ is another extension.

Short Exact Sequence

To generalize the idea of a normal subgroup, we introduce a short exact sequence,

Definition 1

A sequence of groups:

$$1 \xrightarrow{f_1} K \xrightarrow{f_2} G \xrightarrow{f_3} H \xrightarrow{f_4} 1$$

Is called *short exact* if all f_i are homomorphisms and $\text{im}(f_i) = \ker(f_{i+1})$.

G is an extension of H by K if G fits into the above short exact sequence.

Example

The sequence:

$$1 \xrightarrow{f_1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_3} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_4} 1$$

is short exact. To see this, we let $f_2(a) = (a, 0)$, and let $f_3(a, b) = b$, for all $a, b \in \mathbb{Z}/2\mathbb{Z}$.

Main Theorem

Theorem 1.1

There exists a bijection between $H_{\theta}^2(H, K)$ and the set of equivalence classes, E , of extensions which realize the data (H, K, θ) .

As a result, the description of all congruence classes of extensions of H by K can be described as:

$$\bigcup_{\theta \in \text{Hom}(H, \text{Aut}(K))} H_{\theta}^2(H, K)$$

Sketch

- When $G/K \cong H$, it can be noted that H acts upon K .
- This homomorphism is determined by the congruence classes of the extensions of H by K .
- The congruence classes of extensions are able to be partitioned by the elements of $\text{Hom}(H, \text{Aut}(K))$
- For the fixed homomorphism θ , the congruence classes of extensions which induce θ is described by the second cohomology group $H_{\theta}^2(H, K)$.

Transversal

Definition 2

A transversal is a set containing one element from every coset of K . We denote this transversal T . Alternatively, it can also be thought of as a function $\phi : H \rightarrow G$, where $\phi(H) = T$.

Lemma 3

Let T be a transversal of K in G . Then

$$G = \bigcup_{t \in T} tK$$

Example

$K = \mathbb{Z}$ and $H = [0, 1)$, we can see that a transversal of K in \mathbb{R} would be the set $[0, 1)$.

Definition 4

An ordered triple (H, K, θ) is called *data* if K is abelian, H is a group, and $\theta : H \rightarrow \text{Aut}(K)$ is a homomorphism. If G is an extension of H by K , then G *realizes* the data if for every transversal $\phi : H \rightarrow G$:

$$xa = \phi(x) + a - \phi(x).$$

For $a \in K$ and $x \in H$.

Factor Sets

Definition 5

A factor set for the group G is a function $f : H \times H \rightarrow K$ such that for all $x, y \in H$ and some transversal ϕ ,

$$\phi(x) + \phi(y) = f(x, y) + \phi(xy)$$

We call the set of factor sets $Z_{\theta}^2(H, K)$.

Example

Let $K = \mathbb{Z}$ and $H = (0, 1]$, with transversal $T = [0, 1)$, we define $\phi(h) = h$. This means ϕ is a homomorphism, so $f(x, y) = 0$, for all $x, y \in H$.

Construction of G_f

We take this time to provide a construction of a group realizing the data (H, K, θ) and factor set f . We consider the set $K \times H$, along with addition:

$$(k, x) + (l, y) = (k + xl + f(x, y), xy).$$

We denote this group as G_f , and the equivalence class of this group $[G_f]$.

Coboundaries

Definition 6

Given the data, (H, K, θ) , a coboundary is a function $g : H \times H \rightarrow K$, such that for all $x, y \in H$,

$$g(x, y) = xe(y) - e(xy) + e(x)$$

Where $e : H \rightarrow K$ and $e(1) = 0$. The set of all coboundaries is denoted $B_{\theta}^2(H, K)$.

Second Cohomology Group

Definition 7

We define the second cohomology group to be $Z_{\theta}^2(H, K)/B_{\theta}^2(H, K)$. We denote this group as $H_{\theta}^2(H, K)$.

This is not the actual definition of the i th cohomology group, but it will suffice for now.

Main Theorem

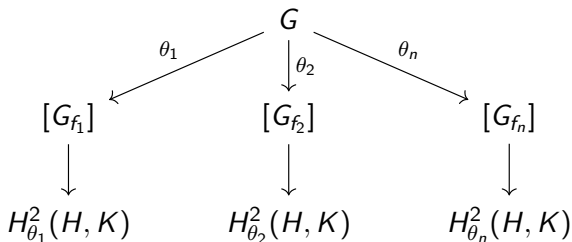
Theorem 1.2

There exists a bijection between $H_\theta^2(H, K)$ and the set of equivalence classes, E , of extensions which realize the data (H, K, θ) .

Proof sketch: We define a function $\rho : H_\theta^2(H, K) \rightarrow E$, as $\rho(f + B_\theta^2(H, K)) = [G_f]$. We prove that ρ is bijective.

Diagram

This is a diagram depicting how we classify the equivalence classes of the extensions of H by K .



Thank you for your attention!

If you have any questions you can message me on discord.