Pell's Equation

Yunhan Tong

July 12, 2024

Introduction

Basic Form

Pell's equation is any Diophantine equation of the form

$$x^2 - dy^2 = 1$$

Basic Form

Pell's equation is any Diophantine equation of the form

$$x^2 - dy^2 = 1$$

- n is a **nonsquare positive integer**.
- We are interested in solutions (x, y) where x and y are both integers, and the term "solution" regarding Pell's equation will always mean an integral solution.

Basic Form

Pell's equation is any Diophantine equation of the form

$$x^2 - dy^2 = 1$$

- n is a **nonsquare positive integer**.
- We are interested in solutions (x, y) where x and y are both integers, and the term "solution" regarding Pell's equation will always mean an integral solution.
- All Pell's equation have **trivial solutions**: $(x, y) = (\pm 1, 0)$
- Solutions which $x > 0$ and $y > 0$ will be called **positive solutions**.
- Every nontrivial solution can be made into a positive solution by changing the sign of x or y.

History

Pell's equation was first studied extensively in India starting with **Brahmagupata**, who described how to use known solutions to create new solutions, and **Bhaskara**, who gave an efficient method for finding a minimal positive solution to Pell's equation. Later, several European mathematicians rediscovered how to solve Pell's equation in 17th century. **Fermat** had contributed to some of the basic theories, and it was **Lagrange** who discovered the complete theory of the equation $x^2 - dy^2 = 1$. **Brounker** had also gave a general method for solving Pell's equation and solve the case $d = 313$. However, Leonhard Euler mistakenly thought this solution was due to **John Pell** who helped with writing a book regarding these equations. As a result, he named the equation after Pell out of accident.

The Existence Of A Nontrival Solution

An interesting and essential property of a Pell's equation is that it always have a nontrivial solution, which is the following theorem:

Theorem (Lagrange)

For all $d \in \mathbb{Z}^+$ that are not perfect squares, the equation $x^2 - dy^2 = 1$ has a nontrivial solution.

An interesting and essential property of a Pell's equation is that it always have a nontrivial solution, which is the following theorem:

Theorem (Lagrange)

For all $d \in \mathbb{Z}^+$ that are not perfect squares, the equation $x^2 - dy^2 = 1$ has a nontrivial solution.

To prove this, we start by the following lemma:

Lemma (Dirichlet's approximation theorem)

For each nonsquare positive integer d, there are infinitely many positive integers x and y such that $|x - y\sqrt{d}| < 1/y$.

Example

We will use an example using the case when $d = 5$ to demonstrate the process in the Lemma. We will give two solutions to $|x - \sqrt{5}y| < 1/y$.

Example

We will use an example using the case when $d = 5$ to demonstrate the process in the Lemma. We will give two solutions to $|x - \sqrt{5}y| < 1/y$. Choose a positive integer m and dividing the interval $[0, 1)$ into m parts. Here we use $m = 10$.

Then among the eleven numbers $0, \sqrt{5}, 2\sqrt{5}, \ldots, 10\sqrt{5}$, there must be two numbers that have fractional parts on the same interval $[i/10, (i+1)/10)$. Listing the fractional parts of $k\sqrt{5}$ for $0 \leq k \leq 10$ rounding to two decimal places, we have:

k	0	1	2	3	4	5	6	7	8	9	10
fractional part	0	.24	.47	.71	.94	.18	.42	.65	.89	.12	.36

Example

Noticing that $k = 2$ and $k = 6$ have fractional parts on the common interval. Let $a = 2$ and $b = 6$, we have: $2\sqrt{5} = 4.47\cdots, 6\sqrt{5} = 13.41\cdots$. Let $x = |\lfloor 2\sqrt{5}\rfloor - \lfloor 6\sqrt{5}\rfloor| = 9$, $y = |b - a| = 4$. Since $a, b \in [0, 1, \ldots, m]$, there is $0 < y \leq m$. Thus,

$$|(2\sqrt{5} - 4) - (6\sqrt{5} - 13)| = |9 - 4\sqrt{5}| \approx 0.05 < \frac{1}{10} < \frac{1}{4}.$$

Therefore, we obtain a positive solution $(x, y) = (9, 4)$.

Example

To get a second pair (x', y') such that $|x' - \sqrt{5}y'| < 1/y'$, choose an m' which satisfies $1/m' < |x - \sqrt{5}y|$. Since $|9 - 4\sqrt{5}| \approx 0.055 > 1/20$, we take $m' = 20$ and seek two numbers from $k\sqrt{5}(0 \le k \le 20)$ that have fractional parts on the same interval $[i/20, (i+1)/20)$. This happens when $k = 1$ and $k = 18$, as $\sqrt{5} = 2.236\cdots$ and $18\sqrt{5} = 40.249\cdots$. So

$$|(\sqrt{5} - 2) - (18 - \sqrt{5}) - 40| = |38 - 17\sqrt{5}| \approx 0.013 < \frac{1}{20} < \frac{1}{17}.$$

This gives another solution $(x', y') = (38, 17)$. In this way, we can get infinite solutions satisfying $|x - y\sqrt{d}| < 1/y$.

From The Fundamental Solution to All Solutions

Fundamental Solution

Definition

For $x^2 - dy^2 = 1$, let the **fundamental solution** (x_0, y_0) be a positive integral solution that minimize the value of $x_0 + \sqrt{d}y_0$.

Fundamental Solution

Definition

For $x^2 - dy^2 = 1$, let the **fundamental solution** (x_0, y_0) be a positive integral solution that minimize the value of $x_0 + \sqrt{d}y_0$.

In fact, this also means that the fundamental solution has the least x and y values among all solutions.

Fundamental Solution

Definition

For $x^2 - dy^2 = 1$, let the **fundamental solution** (x_0, y_0) be a positive integral solution that minimize the value of $x_0 + \sqrt{d}y_0$.

In fact, this also means that the fundamental solution has the least x and y values among all solutions.

Proof.

Conversely, suppose $x_0 > x$, then we have $x_0^2 = dy_0^2 + 1 > x^2 = dy^2 + 1$. Therefore, $y_0 > y$. Now, we have $x_0 + \sqrt{d}y_0 > x + \sqrt{d}y$ which contradicts the minimality of $x_0 + \sqrt{d}y_0$. Thus, x_0 must be smaller or equal to x. Similarly, $y_0 \leq y$. ∎

All Solutions

The following theorem will show how to find all solutions from the fundamental solution.

Theorem

Pell's equations have infinitely many solutions if it has a nontrivial solution (x_0, y_0), and all solutions (x_n, y_n) can be expressed as $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$ for all $n \in \mathbb{Z}^+$.

The proof is consisted of:

1. Show that how that every (x_n, y_n) that satisfies $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$ will be a solution to the Pell's equation.
2. **All** solutions can be expressed as $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$.

All Solutions

Example

$x^2 - 35y^2 = 1$.
The fundamental solution for this Pell's equation is: $(x_0, y_0) = (6, 1)$.
Then, from the method in Theorem we have

$$x_2 + y_2\sqrt{35} = (6 + \sqrt{35})^2 = 71 + 12\sqrt{35}.$$

So we get another solution: $(x_2, y_2) = (71, 12)$. This solution works since

$$71^2 - 35 * 1^2 = 5041 - 5040 = 1.$$

All Solutions

Example

$x^2 - 35y^2 = 1$.
The fundamental solution for this Pell's equation is: $(x_0, y_0) = (6, 1)$.
Then, from the method in Theorem we have

$$x_2 + y_2\sqrt{35} = (6 + \sqrt{35})^2 = 71 + 12\sqrt{35}.$$

So we get another solution: $(x_2, y_2) = (71, 12)$. This solution works since

$$71^2 - 35 * 1^2 = 5041 - 5040 = 1.$$

Using the formula again, we have

$$x_3 + y_3\sqrt{35} = (6 + \sqrt{35})^3 = 846 + 143\sqrt{35}.$$

So $(x_3, y_3) = (846, 143)$. In this way, we can generate all the solutions.

All Solutions

Note: If (x_0, y_0) is the fundamental solution of Pell's equation, then the solutions (x_n, y_n) can be given by the following formula:

$$\begin{cases} x_n = \frac{1}{2}[(x_0 + \sqrt{d}y_0)^n + (x_0 - \sqrt{d}y_0)^n], \\ y_n = \frac{1}{2\sqrt{d}}[(x_0 + \sqrt{d}y_0)^n - (x_0 - \sqrt{d}y_0)^n]. \end{cases} \tag{3.1}$$

However, this formula is too complicated, so we want to find a method that can let us calculate the solutions quickly and easily.

All Solutions

Corollary

All solutions of Pell's equation satisfy the following recursive relationship:

$$\text{For any } n \leq 2 \begin{cases} x_n = 2x_0x_{n-1} - x_{n-2} \\ y_n = 2x_0y_{n-1} - y_{n-2} \end{cases} \tag{3.2}$$

Continued Fractions

Basic Form

Continued fraction is an important topic in Number Theory. It is closely related to Pell's equation and can be used to find the fundamental solution.

Definition

A **continued fraction** is an expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}$$

where a_i and b_i are either real numbers or complex numbers.

Basic Form

Continued fraction is an important topic in Number Theory. It is closely related to Pell's equation and can be used to find the fundamental solution.

Definition

A **continued fraction** is an expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}$$

where a_i and b_i are either real numbers or complex numbers.

- Only a_1 may be zero, and others are all non-negative numbers.
- . If the expression contains finitely many terms, it is called a **finite continued fraction**;
- otherwise it is called an **infinite continued fraction**.

Simple Continued Fraction

If $b_i = 1$ for all i, then the expression is called a **simple** or **regular continued fraction** and has the following form:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

Simple Continued Fraction

If $b_i = 1$ for all i, then the expression is called a **simple** or **regular continued fraction** and has the following form:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

- **Note:**In most cases, it is assumed that the term "continued fraction" refers to the regular form. If the numerator of one of the fractions is not zero, it will be specifically mentioned as **generalized continued fraction**.
- For a simple continued fraction, it can be represented by the abbreviated notation $[a_0; a_1, a_2, \ldots, a_n]$ or $[a_0; a_1, a_2, \ldots]$ depending on whether it terminates.

Periodic Continued Fraction

Definition

Let $[a_0, a_1, a_2, \ldots]$ be a continued fraction such that $a_n = a_{n+l}$ for all sufficiently large n and a fixed positive integer l, then it is **periodic** and l is the length of the period .

We denote this expansion by

$$[a_0, a_1, \ldots, a_{n-1}, \overline{a_n, a_{n+1}, \ldots, a_{n+l}}]$$

Convergent

For $n \leq m, [a_0, a_1, \ldots, a_n]$ is called **nth convergent** to $[a_0, a_1, \ldots, a_m]$. Define two sequences of real numbers, (p_n) and (q_n), recursively as follows:

$$p_0 = a_0, p_1 = a_1 a_0 + 1, p_n = a_n p_{n-1} + p_{n-2} (2 \leq n \leq m)$$

$$q_0 = 1, q_1 = a_1, q_n = a_n q_{n-1} + q_{n-2} (2 \leq n \leq m)$$

Using Continued Fraction to Find Solutions to Pell's Equation

Theorem

For a Pell's equation, let l be the minimal period of the continued fraction of $\sqrt{d}$, then the fundamental solution to this Pell's equation is:

$$(x_1, y_1) = \begin{cases} (p_{l-1}, q_{l-1}) & \text{if } l \text{ is even} \\ (p_{2l-1}, q_{2l-1}) & \text{if } l \text{ is odd} \end{cases} \tag{4.1}$$

Thanks for your listening!