Pell's Equation

Yunhan Tong

July 12, 2024

Yu			

<ロト <回ト < 回ト < 回ト -

2

Introduction

han 1	

3

Pell's equation is any Diophantine equation of the form

 $x^2 - dy^2 = 1$

Yunhan Tong	

3

イロト 不得 トイヨト イヨト

Pell's equation is any Diophantine equation of the form

 $x^2 - dy^2 = 1$

- *n* is a **nonsquare positive integer**.
- We are interested in solutions (x, y) where x and y are both integers, and the term "solution" regarding Pell's equation will always mean an integral solution.

3/22

イロト イポト イヨト イヨト 二日

Pell's equation is any Diophantine equation of the form

 $x^2 - dv^2 = 1$

- *n* is a **nonsquare positive integer**.
- We are interested in solutions (x, y) where x and y are both integers, and the term "solution" regarding Pell's equation will always mean an integral solution.
- All Pell's equation have trivial solutions: $(x, y) = (\pm 1, 0)$
- Solutions which x > 0 and y > 0 will be called **positive solutions**.
- Every nontrivial solution can be made into a positive solution by changing the sign of x or y.

History

Pell's equation was first studied extensively in India starting with **Brahmagupata**, who described how to use known solutions to create new solutions, and **Bhaskara**, who gave an efficient method for finding a minimal positive solution to Pell's equation. Later, several European mathematicians rediscovered how to solve Pell's equation in 17th century. **Fermat** had contributed to some of the basic theories, and it was Lagrange who discovered the complete theory of the equation $x^2 - dy^2 = 1$. Brounker had also gave a general method for solving Pell's equation and solve the case d = 313. However, Leonhard Euler mistakenly thought this solution was due to John Pell who helped with writing a book regarding these equations. As a result, he named the equation after Pell out of accident.

The Existence Of A Nontrival Solution

han 1	

3

An interesting and essential property of a Pell's equation is that it always have a nontrivial solution, which is the following theorem:

Theorem (Lagrange)

For all $d \in \mathbb{Z}^+$ that are not perfect squares, the equation $x^2 - dy^2 = 1$ has a nontrivial solution.

An interesting and essential property of a Pell's equation is that it always have a nontrivial solution, which is the following theorem:

Theorem (Lagrange)

For all $d \in \mathbb{Z}^+$ that are not perfect squares, the equation $x^2 - dy^2 = 1$ has a nontrivial solution.

To prove this, we start by the following lemma:

Lemma (Dirichlet's approximation theorem)

For each nonsquare positive integer d, there are infinitely many positive integers x and y such that $|x - y\sqrt{d}| < 1/y$.

イロト 不得 トイヨト イヨト

We will use an example using the case when d = 5 to demonstrate the process in the Lemma. We will give two solutions to $|x - \sqrt{5}y| < 1/y$.

イロト 不得 トイヨト イヨト

3

We will use an example using the case when d = 5 to demonstrate the process in the Lemma. We will give two solutions to $|x - \sqrt{5}y| < 1/y$. Choose a positive integer *m* and dividing the interval [0, 1) into *m* parts. Here we use m = 10.

Then among the eleven numbers $0, \sqrt{5}, 2\sqrt{5}, \ldots, 10\sqrt{5}$, there must be two numbers that have fractional parts on the same interval [i/10, (i + 1)/10). Listing the fractional parts of $k\sqrt{5}$ for $0 \le k \le 10$ rounding to two decimal places, we have:

k	0	1	2	3	4	5	6	7	8	9	10
fractional part	0	.24	.47	.71	.94	.18	.42	.65	.89	.12	.36

			= 240
Yunhan Tong	Pell's Equation	July 12, 2024	7 / 22

Noticing that k = 2 and k = 6 have fractional parts on the common interval. Let a = 2 and b = 6, we have: $2\sqrt{5} = 4.47 \cdots, 6\sqrt{5} = 13.41 \cdots$. Let $x = |\lfloor 2\sqrt{5} \rfloor - \lfloor 6\sqrt{5} \rfloor| = 9$, y = |b - a| = 4. Since $a, b \in [0, 1, \dots, m]$, there is $0 < y \le m$. Thus,

$$|(2\sqrt{5}-4) - (6\sqrt{5}-13)| = |9-4\sqrt{5}| \approx 0.05 < \frac{1}{10} < \frac{1}{4}$$

Therefore, we obtain a positive solution (x, y) = (9, 4).

8 / 22

イロト イポト イヨト イヨト 二日

To get a second pair (x', y') such that $|x' - \sqrt{5}y'| < 1/y'$, choose an m' which satisfies $1/m' < |x - \sqrt{5}y|$. Since $|9 - 4\sqrt{5}| \approx 0.055 > 1/20$, we take m' = 20 and seek two numbers from $k\sqrt{5}(0 \le k \le 20)$ that have fractional parts on the same interval [i/20, (i+1)/20). This happens when k = 1 and k = 18, as $\sqrt{5} = 2.236 \cdots$ and $18\sqrt{5} = 40.249 \cdots$. So

$$|(\sqrt{5}-2)-(18-\sqrt{5})-40|=|38-17\sqrt{5}|\approx 0.013<rac{1}{20}<rac{1}{17}.$$

This gives another solution (x', y') = (38, 17). In this way, we can get infinite solutions satisfying $|x - y\sqrt{d}| < 1/y$.

イロト 不得下 イヨト イヨト 二日

From The Fundamental Solution to All Solutions

han 1	

э

10 / 22

イロト イヨト イヨト イヨト

Fundamental Solution

Definition

For $x^2 - dy^2 = 1$, let the **fundamental solution** (x_0, y_0) be a positive integral solution that minimize the value of $x_0 + \sqrt{dy_0}$.

3

イロト 不得 トイヨト イヨト

Fundamental Solution

Definition

For $x^2 - dy^2 = 1$, let the **fundamental solution** (x_0, y_0) be a positive integral solution that minimize the value of $x_0 + \sqrt{dy_0}$.

In fact, this also means that the fundamental solution has the least x and y values among all solutions.

イロト 不得 トイヨト イヨト

Fundamental Solution

Definition

For $x^2 - dy^2 = 1$, let the **fundamental solution** (x_0, y_0) be a positive integral solution that minimize the value of $x_0 + \sqrt{dy_0}$.

In fact, this also means that the fundamental solution has the least x and y values among all solutions.

Proof.

Conversely, suppose $x_0 > x$, then we have $x_0^2 = dy_0^2 + 1 > x^2 = dy^2 + 1$. Therefore, $y_0 > y$. Now, we have $x_0 + \sqrt{d}y_0 > x + \sqrt{d}y$ which contradicts the minimality of $x_0 + \sqrt{d}y_0$. Thus, x_0 must be smaller or equal to x. Similarly, $y_0 \le y$.

< ロ > < 同 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < 回 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

The following theorem will show how to find all solutions from the fundamental solution.

Theorem

Pell's equations have infinitely many solutions if it has a nontrivial solution (x_0, y_0) , and all solutions (x_n, y_n) can be expressed as $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$ for all $n \in \mathbb{Z}^+$.

The proof is consisted of:

- Show that how that every (x_n, y_n) that satisfies $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$ will be a solution to the Pell's equation.
- **3** All solutions can be expressed as $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$.

イロト 不得 トイラト イラト 一日

Example

$$x^2 - 35y^2 = 1.$$

The fundamental solution for this Pell's equation is: $(x_0, y_0) = (6, 1)$. Then, from the method in Theorem we have

$$x_2 + y_2\sqrt{35} = (6 + \sqrt{35})^2 = 71 + 12\sqrt{35}.$$

So we get another solution: $(x_2, y_2) = (71, 12)$. This solution works since

$$71^2 - 35 * 1^2 = 5041 - 5040 = 1.$$

Example

$$x^2 - 35y^2 = 1.$$

The fundamental solution for this Pell's equation is: $(x_0, y_0) = (6, 1)$. Then, from the method in Theorem we have

$$x_2 + y_2\sqrt{35} = (6 + \sqrt{35})^2 = 71 + 12\sqrt{35}.$$

So we get another solution: $(x_2, y_2) = (71, 12)$. This solution works since

$$71^2 - 35 * 1^2 = 5041 - 5040 = 1.$$

Using the formula again, we have

$$x_3 + y_3\sqrt{35} = (6 + \sqrt{35})^3 = 846 + 143\sqrt{35}.$$

So $(x_3, y_3) = (846, 143)$. In this way, we can generate all the solutions.

Note: If (x_0, y_0) is the fundamental solution of Pell's equation, then the solutions (x_n, y_n) can be given by the following formula:

$$\begin{cases} x_n = \frac{1}{2} [(x_0 + \sqrt{d}y_0)^n + (x_0 - \sqrt{d}y_0)^n], \\ y_n = \frac{1}{2\sqrt{d}} [(x_0 + \sqrt{d}y_0)^n - (x_0 - \sqrt{d}y_0)^n]. \end{cases}$$
(3.1)

However, this formula is too complicated, so we want to find a method that can let us calculate the solutions quickly and easily.

イロト イポト イヨト イヨト 二日

Corollary

All solutions of Pell's equation satisfy the following recursive relationship:

For any
$$n \leq 2 \begin{cases} x_n = 2x_0x_{n-1} - x_{n-2} \\ y_n = 2x_0y_{n-1} - y_{n-2} \end{cases}$$
 (3.2)

in Tong

Image: A matrix and a matrix

Continued Fractions

han 1	

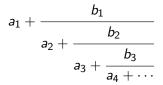
3

イロト イヨト イヨト イヨト

Continued fraction is an important topic in Number Theory. It is closely related to Pell's equation and can be used to find the fundamental solution.

Definition

A continued fraction is an expression of the form



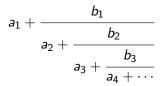
where a_i and b_i are either real numbers or complex numbers.

Yu		

Continued fraction is an important topic in Number Theory. It is closely related to Pell's equation and can be used to find the fundamental solution.

Definition

A continued fraction is an expression of the form



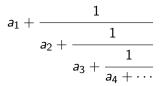
where a_i and b_i are either real numbers or complex numbers.

- Only a_1 may be zero, and others are all non-negative numbers.
- . If the expression contains finitely many terms, it is called a **finite continued fraction**;
- otherwise it is called an infinite continued fraction.

Yunhan Tong

Simple Continued Fraction

If $b_i = 1$ for all *i*, then the expression is called a **simple** or **regular continued fraction** and has the following form:

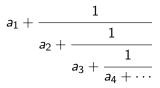


	Ton

イロト イヨト イヨト -

Simple Continued Fraction

If $b_i = 1$ for all *i*, then the expression is called a **simple** or **regular continued fraction** and has the following form:



- Note: In most cases, it is assumed that the term "continued fraction" refers to the regular form. If the numerator of one of the fractions is not zero, it will be specifically mentioned as generalized continued fraction.
- For a simple continued fraction, it can be represented by the abbreviated notation [a₀; a₁, a₂,..., a_n] or [a₀; a₁, a₂,...] depending on whether it terminates.

Yunhan Tong

Periodic Continued Fraction

Definition

Let $[a_0, a_1, a_2, ...]$ be a continued fraction such that $a_n = a_{n+1}$ for all sufficiently large n and a fixed positive integer l, then it is **periodic** and l is the length of the period .

We denote this expansion by

$$[a_0, a_1, \ldots, a_{n-1}, \overline{a_n, a_{n+1}, \ldots, a_{n+l}}]$$

Convergent

For $n \le m, [a_0, a_1, \dots, a_n]$ is called **nth convergent** to $[a_0, a_1, \dots, a_m]$. Define two sequences of real numbers, (p_n) and (q_n) , recursively as follows:

$$p_0 = a_0, p_1 = a_1a_0 + 1, p_n = a_np_{n-1} + p_{n-2}(2 \le n \le m)$$

 $q_0 = 1, q_1 = a_1, q_n = a_nq_{n-1} + q_{n-2}(2 \le n \le m)$

イロト 不得下 イヨト イヨト 二日

Using Continued Fraction to Find Solutions to Pell's Equation

Theorem

For a Pell's equation, let I be the minimal period of the continued fraction of \sqrt{d} , then the fundamental solution to this Pell's equation is:

$$(x_1, y_1) = \begin{cases} (p_{l-1}, q_{l-1}) & \text{if } l \text{ is even} \\ (p_{2l-1}, q_{2l-1}) & \text{if } l \text{ is odd} \end{cases}$$
(4.1)

han 🛛	

Thanks for your listening!

han 1	

3

22 / 22

・ロト ・ 日 ト ・ 日 ト ・ 日 ト