Pell's Equation

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Abstract

This is an expository paper about Pell's equation, including the proof of the existence of a nontrival solution, the way to solve all solutions based on fundamental solution, and the way to find the fundamental solution. Some basics about continued fractions will also be introduced in this paper, serving as a useful tool for solving the fundamental solution of a Pell's equation. Finally, we will talk about the application of Pell's equation.

1 Introduction

Pell's equation is any Diophantine equation of the form

$$x^2 - dy^2 = 1 \tag{*}$$

where n is a **nonsquare positive integer**. The equation was first studied in the case $x^2 - dy^2 = 1$ because early mathematicians, upon discovering that $\sqrt{2}$ is irrational, realized that one cannot solve the equation $x^2 - dy^2 = 0$, and started to attempt solving the "next best things". Pell's equation was first studied extensively in India starting with **Brahmagupata**, who described how to use known solutions to create new solutions, and **Bhaskara**, who gave an efficient method for finding a minimal positive solution to Pell's equation. Later, several European mathematicians rediscovered how to solve Pell's equation in 17th century. **Fermat** had contributed to some of the basic theories, and it was **Lagrange** who discovered the complete theory of the equation $x^2 - dy^2 = 1$. **Brounker** had also gave a general method for solving Pell's equation and solve the case $d = 313$. However, Leonhard Euler mistakenly thought this solution was due to **John Pell** who helped with writing a book regarding these equations. As a result, he named the equation after Pell out of accident.

For equation (*), we are interested in solutions (x, y) where x and y are both integers, and the term "solution" regarding Pell's equation will always mean an integral solution. There are **trivial solutions** $(x, y) = (\pm 1, 0)$ that works for all (*). They are the only solutions when d is negative (that's why we are only interested in the cases for $d > 0$). The other solutions, which we will be studying in this paper, will be called the **nontrivial solutions**. Solutions which $x > 0$ and $y > 0$ will be called **positive solutions**. Due to the property of square numbers, every nontrivial solution can be made into a positive solution by changing the sign of x or y. For the sake of this paper, we will only talk about the ways to find the positive solutions.

In addition, we don't consider the case when d is a square since if $d = c^2$ with $c \in \mathbb{Z}$, there is $x^2 - (cy)^2 = 1$. The only perfect squares that differ by 1 are 0 and 1, so $x^2 = 1$ and $(cy)^2 = 0$, which gives $x = \pm 1$ and $y = 0$. Thus, this kind of Pell's equation only has trivial solutions.

2 The Existence of a Nontrivial Solution

For Pell's equation (*), a trivial solution is the obvious solutions $(x, y) = (\pm 1, 0)$ which works for all $x^2 - dy^2 = 1$. Our goal is to find the nontrivial solutions. Before introducing the methods to solve it, we will first prove that this kind of solution always exists for a valid Pell's equation. The starting point is the following lemma:

Lemma 2.1 (Dirichlet's approximation theorem)**.** *For each nonsquare positive integer d, there are infinitely many positive integers x and y such that $|x - y\sqrt{d}| < 1/y$.*

Proof. Construct a set of $m + 1$ numbers:

$$0, \sqrt{d}, 2\sqrt{d}, \ldots, m\sqrt{d}$$

Each of these number will have a fractional part belonging to the interval $[0, 1)$. View $[0, 1)$ as m half-open intervals: $[0, 1/m), [1/m, 2/m), \ldots, [(m-1)/m, 1)$. By the pigeonhole principle, two of the $m + 1$ numbers, say $a\sqrt{d}$ and $b\sqrt{d}$ with $a < b$, must have fractional parts in the same interval. Suppose that

$$a\sqrt{d} = A + \epsilon, \quad b\sqrt{d} = B + \delta.$$

where $A, B \in \mathbb{Z}$ and ϵ, δ belongs to the same interval $[i/m, (i + 1)/m)$ for a certain m. Therefore,

$$|\epsilon - \delta| < \frac{1}{m}$$
$$\Longrightarrow |(a\sqrt{d} - A) - (b\sqrt{d} - B)| < \frac{1}{m}$$
$$\Longrightarrow |(B - A) - (b - a)\sqrt{d}| < \frac{1}{m}.$$

Set $x = B - A$ and $y = b - a$ where x and y are integers. Since $a, b \in [0, 1, \ldots, m]$, there is $0 < y \leq m$. Substituting this to the inequality above we get

$$|x - y\sqrt{d}| < \frac{1}{m} \leq \frac{1}{y}.$$

Since it can be derived that $|x-y\sqrt{d}| < 1$, we have $x > y\sqrt{d}-1 \geq \sqrt{d}-1 > 0$, so x is positive. Thus, we successfully create one pair of positive integers (x, y) such that $|x - y\sqrt{d}| < 1/y$. To find another pair, choose a positive integer m' such that $1/m' < |x - y\sqrt{d}|$. There is always such an m' because $x - \sqrt{d}y \neq 0$ (since $\sqrt{d}$ is irrational). run through the arguments above replacing m with m', which will result in x' and y' that satisfy $|x' - y'\sqrt{d}| < 1/y'$. (x', y') is obviously different from (x, y) because of the following relationship:

$$|x - y\sqrt{d}| > \frac{1}{m'} > |x' - y'\sqrt{d}|.$$

Furthermore, by repeating the operation we can get

$$|x - y\sqrt{d}| > \frac{1}{m'} > |x' - y'\sqrt{d}| > m'' > |x'' - y''\sqrt{d}| > \cdots$$

which can be extended unlimitedly. In this way, we can obtain infinitely many pairs of x and y satisfying $|x - y\sqrt{d}| < 1/y$. ■

Example. We will use an example using the case when $d = 5$ to demonstrate the process in Lemma 2.1. We will give two solutions to $|x - \sqrt{5}y| < 1/y$.

Taking $m = 10$, then among the eleven numbers $0, \sqrt{5}, 2\sqrt{5}, \ldots, 10\sqrt{5}$, there are at least two numbers that have fractional parts on the same interval $[i/10, (i+1)/10)$. Listing the fractional parts of $k\sqrt{5}$ for $0 \le k \le 10$ rounding to two decimal places, we have:

k	0	1	2	3	4	5	6	7	8	9	10
Fractional part of $k\sqrt{5}$	0	.24	.47	.71	.94	.18	.42	.65	.89	.12	.36

Noticing that $k = 2$ and $k = 6$ have fractional parts on the common interval. Let $a = 2$ and $b = 6$, we have: $2\sqrt{5} = 4.47\cdots, 6\sqrt{5} = 13.41\cdots$. Thus,

$$|(2\sqrt{5} - 4) - (6\sqrt{5} - 13)| = |9 - 4\sqrt{5}| \approx 0.05 < \frac{1}{10} < \frac{1}{4}.$$

Therefore, we obtain a positive solution $(x, y) = (9, 4)$.

To get a second pair (x', y') such that $|x' - \sqrt{5}y'| < 1/y'$, choose an m' which satisfies $1/m' < |x - \sqrt{5}y|$. Since $|9 - 4\sqrt{5}| \approx 0.055 > 1/20$, we take $m' = 20$ and seek two numbers from $k\sqrt{5}(0 \le k \le 20)$ that have fractional parts on the same interval $[i/20, (i+1)/20)$. This happens when $k = 1$ and $k = 18$, as $\sqrt{5} = 2.236\cdots$ and $18\sqrt{5} = 40.249\cdots$. So

$$|(\sqrt{5} - 2) - (18 - \sqrt{5}) - 40| = |38 - 17\sqrt{5}| \approx 0.013 < \frac{1}{20} < \frac{1}{17}.$$

This gives another solution $(x', y') = (38, 17)$.

Lemma 2.2. *For each positive integers x and y satisfying $|x - \sqrt{d}y| < 1/y$ there is $|x^2 - \sqrt{d}y^2| < 1 + 2\sqrt{d}$.*

Proof. First, we will show that x has the upper bound as follows:

$$x = x - y\sqrt{d} + y\sqrt{d} \le |x - y\sqrt{d}| + y\sqrt{d} < \frac{1}{y} + y\sqrt{d} \le 1 + y\sqrt{d}.$$

Thus, we have

$$|x^2 - dy^2| = (x + y\sqrt{d})|x - y\sqrt{d}| < (1 + y\sqrt{d} + y\sqrt{d})\frac{1}{y} = \frac{1}{y} + 2\sqrt{d} \le 1 + 2\sqrt{d}.$$

■

Theorem 2.3 (Lagrange)**.** *For all $d \in \mathbb{Z}^+$ that are not perfect squares, the equation $x^2 - dy^2 = 1$ has a nontrivial solution.*

Proof. From Lemma 2.3, $|x^2 - dy^2| < 1 + 2\sqrt{d}$ for infinitely many pairs of positive integers (x, y). Since there are only finitely many integers between $-1 - 2\sqrt{d}$ and $1 + 2\sqrt{d}$, by the pigeonhole principle, there exists an integer M with $|M| < 1 + 2\sqrt{d}$ such that

$$x^2 - dy^2 = M \tag{2.1}$$

for infinitely many positive integer pairs (x, y), and $M \neq 0$ since $\sqrt{d}$ is irrational. For x and y satisfying equation (2.1), reduce x and y modulo $|M|$. By the pigeonhole principle, there must be a repetition for the infinitely many pairs (x mod $|M|$, y mod$|M|$) since there are only finitely many integers mod $|M|$. Therefore, there are distinct positive integer solutions (x_1, y_1) and (x_2, y_2) to equation (2.1) that satisfy $x_1 \equiv x_2 \mod |M|$ and $y_1 \equiv y_2 \mod |M|$. Write $x_1 = x_2 + Mk$, $y_1 = y_2 + Ml$ ($k, l \in \mathbb{Z}$), then

$$x_1 + y_1\sqrt{d} = x_2 + y_2\sqrt{d} + M(k + l\sqrt{d})$$

$$x_1 - y_1\sqrt{d} = x_2 - y_2\sqrt{d} + M(k - l\sqrt{d})$$

Substituting $M = x_2^2 - dy_2^2 = (x_2 + \sqrt{d}y_2)(x_2 - \sqrt{d}y_2)$, we get

$$x_1 + y_1\sqrt{d} = (x_2 + y_2\sqrt{d})(1 + (x_2 - \sqrt{d}y_2)(k + l\sqrt{d})), \tag{2.2}$$

$$x_1 - y_1\sqrt{d} = (x_2 - y_2\sqrt{d})(1 + (x_2 + \sqrt{d}y_2)(k - l\sqrt{d})). \tag{2.3}$$

Combine like terms in the second factor on the right side of (2.2) to rewrite it as $x + y\sqrt{d}$, then that of (2.3) will be $x - y\sqrt{d}$. So we have

$$x_1 + y_1\sqrt{d} = (x_2 + y_2\sqrt{d})(x + y\sqrt{d}), \tag{2.4}$$

$$x_1 - y_1\sqrt{d} = (x_2 - y_2\sqrt{d})(x - y\sqrt{d}). \tag{2.5}$$

Multiplying the last two equations together, there is $M = M(x^2 - dy^2)$. Therefore, we get an integral solution to the equation $x^2 - dy^2 = 1$. To show that this solution is nontrivial, that is, $(x, y) \neq (\pm 1, 0)$, assume otherwise. If $(x, y) = (1, 0)$, we will notice that $x_1 = x_2$ and $y_1 = y_2$ when substituting them to either (2.4) or (2.5). This contradicts the fact that (x_1, y_1) and (x_2, y_2) are different. Similarly, if $(x, y) = (-1, 0)$ then $x_1 = -x_2$, which contradicts that x_1 and x_2 are both positive. ■

3 From The Fundamental Solution to All Solutions

In last section, we discover that there is always a nontrivial solution to a Pell's equation, which will be the hunting license for us solve the equation. In fact, to find all of its solutions, our starting point is to find a solution called the **fundamental solution**.

Definition 3.1. For equation (*), let the **fundamental solution** (x_0, y_0) be a positive integral solution that minimize the value of $x_0 + \sqrt{d}y_0$.

In fact, this also means that the fundamental solution has the least x and y values among all solutions, which can be shown by the following lemma:

Lemma 3.2. *If a Pell's equation (*) has the fundamental solution (x_0, y_0), then every solution (x, y) of this equation must satisfy $x_0 \le x, y_0 \le y$.*

Proof. Conversely, suppose $x_0 > x$, then we have $x_0^2 = dy_0^2 + 1 > x^2 = dy^2 + 1$. Therefore, $y_0 > y$. Now, we have $x_0 + \sqrt{d}y_0 > x + \sqrt{d}y$ which contradicts the minimality of $x_0 + \sqrt{d}y_0$. Thus, x_0 must be smaller or equal to x. Similarly, $y_0 \le y$. ■

Lemma 3.3. *For integers x and y, if $x^2 - dy^2 = 1$ and $x + y\sqrt{d} > 1$ then x and y are both positive integers.*

Proof. The crucial point is that $1/(x + \sqrt{d}y) = x - \sqrt{d}y$ when $x^2 - dy^2 = 1$. Therefore

$$x + \sqrt{d}y > 1 > x - \sqrt{d}y > 0.$$

So $2x > 1 \implies x > 0 \implies x \in \mathbb{N}^+$. Additionally, from $x - \sqrt{d}y < 1 < x + \sqrt{d}y$ we have $\sqrt{d}y > 0 \implies y \in \mathbb{N}^+$. ■

Theorem 3.4. *Pell's equations have infinitely many solutions if it has a nontrivial solution (x_0, y_0), and all solutions (x_n, y_n) can be expressed as $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$ for all $n \in \mathbb{Z}^+$.*

Proof. We will first show that every (x_n, y_n) that satisfies $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$ will be a valid solution to (*). By the binomial theorem, if

$$x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n \tag{3.1}$$

then it must also satisfy that

$$x_n - \sqrt{d}y_n = (x_0 - \sqrt{d}y_0)^n. \tag{3.2}$$

Multiplying both sides of (3.1) and (3.2), there is

$$x_n^2 - dy_n^2 = (x_0 + \sqrt{d}y_0)^n (x_0 - \sqrt{d}y_0)^n = (x_0^2 - dy_0^2)^n = 1.$$

Therefore, (x_n, y_n) is a positive solution to (*). Next, we want to prove that *all* solutions can be expressed as $x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$, which means no other solutions will be found aside from using this formula.

Conversely, assume that there exists a solution (x, y) that cannot be expressed by equation (3.1), namely $x + \sqrt{d}y \ne (x_0 + \sqrt{d}y)^n$. Then there is a specific positive integer r that satisfies

$$(x_0 + \sqrt{d}y_0)^r < x + \sqrt{d}y < (x_0 + \sqrt{d}y_0)^{r+1}$$

$$\iff 1 < \frac{x + \sqrt{d}y}{(x_0 + \sqrt{d}y_0)^r} < x_0 + \sqrt{d}y_0. \tag{3.3}$$

Since

$$\frac{1}{(x_0+\sqrt{d}y_0)^r} = \frac{(x_0-\sqrt{d}y_0)^r}{(x_0+\sqrt{d}y_0)^r(x_0-\sqrt{d}y_0)^r}$$
$$= \frac{(x_0-\sqrt{d}y_0)^r}{(x_0^2-dy_0^2)^r}$$
$$= (x_0-\sqrt{d}y_0)^r,$$

substitute it to the second item of (3.3) we have

$$1 < (x+\sqrt{d}y)(x_0-\sqrt{d}y_0)^r < x_0+\sqrt{d}y_0. \tag{3.4}$$

Suppose that $X+\sqrt{d}Y = (x+\sqrt{d}y)(x_0-\sqrt{d}y_0)^r$ where X and Y are both integers and Y. Consequently, we also have $X^2-dY^2 = (x-\sqrt{d}y)(x_0+\sqrt{d}y_0)^r$. Multiplying them together we will get

$$X^2-dY^2 = (X+\sqrt{d}Y)(X-\sqrt{d}Y) = (x+\sqrt{d}y)(x_0-\sqrt{d}y_0)^r(x-\sqrt{d}y)(x_0+\sqrt{d}y_0)^r = 1$$

We can also prove that X and Y must be positive integers in this case using Lemma 3.3.

Thus, (X,Y) is a positive solution of (*) with $1 < X+\sqrt{d}Y < x_0+\sqrt{d}y_0$ according to 3.4. However, this contradicts the minimality of $x_0+\sqrt{d}y_0$, so the solution is invalid.

Therefore, all solutions can be and must be generated from $x_n+\sqrt{d}y_n = (x_0+\sqrt{d}y_0)^n$. ∎

Example. $x^2-35y^2=1$.

The fundamental solution for this Pell's equation is: $(x_0,y_0)=(6,1)$. Then, from the method in Theorem 3.4 we have

$$x_2+y_2\sqrt{35} = (6+\sqrt{35})^2 = 71+12\sqrt{35}.$$

So we get another solution: $(x_2,y_2)=(71,12)$. This solution works since

$$71^2-35*1^2 = 5041-5040 = 1.$$

Using the formula again, we have

$$x_3+y_3\sqrt{35} = (6+\sqrt{35})^3 = 846+143\sqrt{35}.$$

So $(x_3,y_3)=(846,143)$. To check it, we will get

$$846^2-35*143^2 = 715716-715715 = 1.$$

In this way, we can generate all the solutions of $x^2-35y^2=1$.

Note: If (x_0,y_0) is the fundamental solution of (*), then the solutions (x_n,y_n) can be given by the following formula:

$$\begin{cases} x_n = \frac{1}{2}[(x_0+\sqrt{d}y_0)^n+(x_0-\sqrt{d}y_0)^n], \\ y_n = \frac{1}{2\sqrt{d}}[(x_0+\sqrt{d}y_0)^n-(x_0-\sqrt{d}y_0)^n]. \end{cases} \tag{3.5}$$

However, this formula is too complicated, so we want to find a method that can let us calculate the solutions quickly and easily.

Corollary 3.5. *All solutions of Pell's equation (*) satisfy the following recursive relationship:*

$$\text{For any } n \ \leq\ 2 \begin{cases} x_n = 2x_0 x_{n-1} - x_{n-2} \\ y_n = 2x_0 y_{n-1} - y_{n-2} \end{cases} \tag{3.6}$$

Proof. The proof is rather complex and involving a lot of manipulations. We start by transforming the expression of $x_n + \sqrt{d}y_n$:

$$\begin{aligned} x_n + \sqrt{d}y_n &= (x_0 + \sqrt{d}y_0)^n \\ &= (x_0 + \sqrt{d}y_0)^{n-1}(x_0 + \sqrt{d}y_0) \\ &= (x_{n-1} + \sqrt{d}y_{n-1})(x_0 + \sqrt{d}y_0) \\ &= x_0 x_{n-1} + x_{n-1}\sqrt{d}y_0 + x_0\sqrt{d}y_{n-1} + dy_0 y_{n-1}. \end{aligned} \tag{3.7}$$

Thus,

$$x_n = x_0 x_{n-1} + dy_0 y_{n-1}, \tag{3.8}$$

$$y_n = x_0 y_{n-1} + y_0 x_{n-1}. \tag{3.9}$$

Replace the n's in equation (3.8) with $n-1$, then

$$x_{n-1} = x_0 x_{n-2} + dy_0 y_{n-2}. \tag{3.10}$$

Multiply x_0 on both sides, then

$$x_0 x_{n-1} = x_0^2 x_{n-2} + dx_0 y_0 y_{n-2}. \tag{3.11}$$

Subtracting equation (3.8) by equation (3.11) result in

$$x_n = 2x_0 x_{n-1} - x_0^2 x_{n-2} + dy_0(y_{n-1} - x_0 y_{n-2}). \tag{3.12}$$

Replace the the n's in equation (3.9) with $n-1$, then

$$y_{n-1} = x_0 y_{n-2} + y_0 x_{n-2}. \tag{3.13}$$

Substituting equation (3.13) to equation (3.12), we have

$$\begin{aligned} x_n &= 2x_0 x_{n-1} - x_0^2 x_{n-2} + dy_0^2 x_{n-2} \\ &= 2x_0 x_{n-1} - x_{n-2}(x_0^2 - dy_0^2) \\ &= 2x_0 x_{n-1} - x_{n-2}. \end{aligned}$$

So the recursive relationship for x is proven. To prove that for y, substitute equation (3.10) to equation (3.9):

$$y_n = x_0 y_{n-1} + x_0 y_0 x_{n-2} + dy_0^2 y_{n-2}.$$

Furthermore, since $dy_0^2 = x_0^2 - 1$, we have

$$\begin{aligned} y_n &= x_0 y_{n-1} + x_0 y_0 x_{n-2} + (x_0^2 - 1)y_{n-2} \\ &= x_0 y_{n-1} + x_0 y_0 x_{n-2} + x_0^2 y_{n-2} - y_{n-2} \\ &= x_0 y_{n-1} + x_0(y_0 x_{n-2} + x_0 y_{n-2}) - y_{n-2} \end{aligned}$$

Finally, from equation (3.13),

$$y_n = 2x_0 y_{n-1} - y_{n-2}.$$

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4 Continued Fractions

From Section 3, we learn how all solutions of a Pell's equation is derived from its fundamental solution. However, the problem is: *how to find a fundamental solution?*
An elementary method to find a nontrivial solution of $x^2 - dy^2 = 1$ is through **trial and error**. Rewrite the equation to $x^2 = dy^2 + 1$ and set $y = 1, 2, 3, \ldots$ until x^2 can be a perfect square. This effectively produce a fundamental solution(x, y). However, the numbers can be prohibitively large for this method. This leads to our useful tool for finding the fundamental solution — continued fractions. This section will introduce some basics of continued fractions, and at last, will explain how to use it to solve a Pell's equation.

Definition 4.1. A **continued fraction** is an expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}$$

where a_i and b_i are either real numbers or complex numbers. Only a_1 may be zero, and others are all non-negative numbers. If the expression contains finitely many terms, it is called a **finite continued fraction**;otherwise it is called an **infinite continued fraction**. The number a_i is called the **partial quotients**.

If $b_i = 1$ for all i, then the expression is called a **simple** or **regular continued fraction** and has the following form:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

In most cases, it is assumed that the term "continued fraction" refers to the regular form. If the numerator of one of the fractions is not zero, it will be specifically mentioned as **generalized continued fraction**. For a simple continued fraction, it can be represented by the abbreviated notation $[a_0; a_1, a_2, \ldots, a_n]$ or $[a_0; a_1, a_2, \ldots]$ depending on whether it terminates.

Some examples will be given to show how to write the continued fraction expansion for a fractional number:

Example. Consider $\frac{13}{5}$, which we write as a whole number plus a remainder:

$$\frac{13}{5} = 2 + \frac{3}{5}.$$

Now, consider the **reciprocal** of the remainder, which is $\frac{5}{3}$, and repeat the first step. Thus, we have

$$\frac{5}{3} = 1 + \frac{2}{3}.$$

Repeat again with the reciprocal of $\frac{2}{3}$:

$$\frac{3}{2} = 1 + \frac{1}{2}.$$

This time, the reciprocal of the remainder is 2, which is a rational with no remainder. So we stop here and get the following expression:

$$2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}.$$

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Definition 4.2. For $n \leq m, [a_0, a_1, \ldots, a_n]$ is called **nth convergent** to $[a_0, a_1, \ldots, a_m]$. Define two sequences of real numbers, (p_n) and (q_n), recursively as follows:

$$p_0 = a_0, p_1 = a_1 a_0 + 1, p_n = a_n p_{n-1} + p_{n-2} (2 \leq n \leq m)$$

$$q_0 = 1, q_1 = a_1, q_n = a_n q_{n-1} + q_{n-2} (2 \leq n \leq m)$$

Theorem 4.3. *Let* $[a_0, a_1, \ldots, a_m]$ *be a continued fraction. Then, for* $0 \leq n \leq m, \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$.

Proof. The proof can be done by induction.
For $n = 0$,

$$\frac{p_0}{q_0} = a_0$$

For $n = 1$,

$$\begin{aligned} \frac{p_1}{q_1} &= \frac{a_1 a_0 + 1}{a_1} \\ &= a_0 + \frac{1}{a_1} \\ &= [a_0, a_1] \end{aligned}$$

Now, suppose the theorem holds for $n - 1$. We have that

$$\begin{aligned} [a_0, \ldots, a_n] &= [a_0, a_1, \ldots, a_{n-1} + \frac{1}{a_n}] \\ &= \frac{(a_{n-1} + \frac{1}{a_n}) p_{n-2} + p_{n-3}}{(a_{n-1} + \frac{1}{a_n}) q_{n-2} + q_{n-3}} \\ &= \frac{(a_n a_{n-1} + 1) p_{n-2} + a_n p_{n-3}}{(a_n a_{n-1} + 1) q_{n-2} + a_n q_{n-3}} \\ &= \frac{a_n (a_{n-1} p_{n-2} + p_{n-3}) + p_{n-2}}{a_n (a_{n-1} q_{n-2} + p_{n-3}) + q_{n-2}} \\ &= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \\ &= \frac{p_n}{q_n} \end{aligned}$$

So the theorem holds for each $n \geq 0$. ■

Example. For the continued fraction

$$\frac{19}{51} = [0; 2, 1, 2, 6]$$

$p_0 = 0, q_0 = 1 \Longrightarrow C_0 = \frac{p_0}{q_0} = 0.$
$p_1 = 1, q_1 = 2 \Longrightarrow C_1 = \frac{p_1}{q_1} = \frac{1}{2}.$
$p_2 = 1 * 1 + 0 = 1, q_2 = 1 * 2 + 1 = 3 \Longrightarrow C_3 = \frac{p_2}{q_2} = \frac{1}{3}.$
$p_3 = 2 * 1 + 1 = 3, q_3 = 2 * 3 + 2 = 8 \Longrightarrow C_0 = \frac{p_3}{q_3} = \frac{3}{8}.$
$p_4 = 6 * 3 + 1 = 19, q_4 = 6 * 8 + 3 = 51 \Longrightarrow C_0 = \frac{p_4}{q_4} = \frac{19}{51}.$

Definition 4.4. Let $[a_0, a_1, a_2, \ldots]$ be a continued fraction such that $a_n = a_{n+l}$ for all sufficiently large n and a fixed positive integer l, then it is **periodic** and l is the length of the period .
We denote this expansion by

$$[a_0, a_1, \ldots, a_{n-1}, \overline{a_n, a_{n+1}, \ldots, a_{n+l}}]$$

Example. Converting the irrational number $\sqrt{5}$ to its continued fraction expansion:

$$\sqrt{5} = 2 + \frac{1}{\frac{1}{\sqrt{5}-2}} = 2 + \frac{1}{\frac{\sqrt{5}+2}{1}} = 2 + \frac{1}{4 + (\sqrt{5} - 2)} = 2 + \frac{1}{4 + \frac{1}{\frac{1}{\sqrt{5}-2}}}.$$

Since the $\frac{1}{\sqrt{5}-2}$ has appeared before, the continued fraction will continue to split 4's. So the result is that $\sqrt{5} = [2; \overline{4}]$.

Theorem 4.5. *For a Pell's equation(*), let l be the minimal period of the continued fraction of $\sqrt{d}$, then the fundamental solution to this Pell's equation is:*

$$(x_1, y_1) = \begin{cases} (p_{l-1}, q_{l-1}) & \text{if } l \text{ is even} \\ (p_{2l-1}, q_{2l-1}) & \text{if } l \text{ is odd} \end{cases} \tag{4.1}$$

5 Applications of Pell's equation

A very well-known application of Pell's equation is its connection with **triangular-square numbers**.

Definition 5.1. A positive integer n is called **triangular** if n dots can be arranged to look like an equilateral triangle, which means it is of the form:

$$\sum_{k=1}^{m} k = \frac{1}{2}m(m + 1)$$

for some $m \in \mathbb{N}$. For example, the first four triangular numbers are 1, 3, 6, and 10.

Definition 5.2. Triangular-square numbers are integers which are simultaneously perfect squares and triangular. For example, 36 is a triangular-square number.

Theorem 5.3. *Triangular-square numbers correspond to solutions of* $x^2 - dy^2 = 1$ *for positive integers* x *and* y*.*

Proof. By the definition of a triangular-square number, we know that it must be n^2 for some $n \in \mathbb{N}$ and $\frac{1}{2}m(m+1)$ for some $m \in \mathbb{N}$. Therefore,

$$\begin{aligned} & \frac{m(m+1)}{2} = n^2 \\ \iff \quad & m^2 + m = 2n^2 \\ \iff \quad & (m + \frac{1}{2})^2 - \frac{1}{4} = 2n^2 \\ \iff \quad & (2m+1)^2 - 1 = 2(2n)^2 \\ \iff \quad & (2m+1)^2 - 2(2m)^2 = 1. \end{aligned} \tag{5.1}$$

Since every step is reversible, finding a triangular-square number is equivalent to solving $x^2 - 2y^2 = 1$ in positive integers x and y where $x = 2m+1$ is odd and $y = 2m$ is even. ■

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