Quaternions: Algebra and Applications to Aerospace

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Introduction

What is a Quaternion?

Definition

A quaternion is a four-dimensional vector that can be represented algebraically as $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ and i, j, k are the standard orthonormal basis for $\mathbb{R}^3$ that have the following properties:

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k = -ji$$

$$jk = i = -kj$$

$$ki = k = -ik$$

We can also write a quaternion q as $q = q_0 + \mathbf{q}$ where $q_0 = a$ (the scalar part) and $\mathbf{q} = bi + cj + dk$ (the vector part). It can also be written trigonometrically as:

$$q = \cos(\theta) + u \sin(\theta)$$

Where u is a vector associated with q and θ is the angle associated with q.

Quaternion Algebra

Definition

Quaternion addition and subtraction are componentwise. You add or subtract corresponding terms to each other.

Example

Define the first quaternion as $q_1 = a + bi + cj + dk$ and the second as $q_2 = w + xi + yj + zk$. Their sum is $q_1 + q_2 = a + w + i(b + x) + j(c + y) + k(d + z)$. Their difference is $q_1 - q_2 = a - w + i(b - x) + j(c - y) + k(d - z)$

Quaternion Algebra

Definition

Quaternion multiplication can be approached in two ways: the distributive property and a formula derived by their scalar and cross product. Both methods are done with the properties of multiplication of $i, j,$ and k vectors in mind. The formula for the second way mentioned above when two quaternions p and q are multiplied is

$$pq = p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}$$

where $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$.

Definition

To divide two quaternions, say q_1 and q_2, you would multiply q_1 by the multiplicative inverse of q_2.

Quaternion Algebra

The way we have defined quaternions thus far makes it difficult to get a multiplicative inverse, so we shall also define a quaternion in matrix form for ease of calculation. Any quaternion with coefficients a, b, c, d can be written as the following matrix:

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

This makes it easier to divide quaternions since the multiplicative inverse of a matrix is fairly trivial.

Quaternion Algebra

Definition

Quaternions have complex conjugates that are defined as follows: given a quaternion $q = a + bi + cj + dk$, its complex conjugate is $q^* = a - bi - cj - dk$. This can also be written as the quaternion $q = q_0 + \mathbf{q}$ and its conjugate $q^* = q_0 - \mathbf{q}$.

Definition

A norm is the length of an object. The norm of a quaternion q is $N(q)$ or $|q|$ and can be found by calculating $\sqrt{q^*q}$. A property we can derive from this is the fact that $N^2(pq) = N^2(p)N^2(q)$ where p and q are quaternions.

Definition

The multiplicative inverse of a quaternion can be found by converting it into a matrix and taking its multiplicative inverse. It can also be found using this formula:

$$q^{-1} = \frac{q^*}{|q|^2}$$

Rotation Operator

Exposition

Quaternions are considered widely applicable mainly due to one property: rotations. Quaternions, despite being four dimensional objects, can help model three-dimensional rotation in:

- Phone orientation
- Aircraft orientation
- Celestial movement
- Computer graphics
- Video game development

And so much more. But how?

Goal

The goal in this section is to derive the potential rotation operators that could allow a quaternion to rotate something in $\mathbb{R}^3$. To start, we introduce this definition.

Definition

Pure quaternions are quaternions who have the scalar part of their expression set to 0. They are essentially vectors in $\mathbb{R}^3$ and can be treated as such. Say that Q_0 is the set of all pure quaternions and it is a subset of Q, the set of all quaternions. We establish a correspondence between any vector $v \in \mathbb{R}^3$ and the pure quaternion $q = 0 + v \in Q_0$. We write this correspondence as $v \in \mathbb{R}^3 \leftrightarrow q = 0 + v \in Q_0 \subset Q$.

Derivation of the Rotation Operator: Using One Quaternion

Here, we will examine if one quaternion is enough to produce a rotation in $\mathbb{R}^3$. Say that we have a vector $v \in \mathbb{R}^3$ and we have an image $w \in \mathbb{R}^3$ and a quaternion $q \in Q$. Given these conditions, we will examine $w = qv$.

$$qv = -\mathbf{q} \cdot v + q_0 v + \mathbf{q} \times v$$

This means that our result does not necessarily correspond to a vector in $\mathbb{R}^3$ unless $q \cdot v = 0$ and q and v are orthogonal. This means that a quaternion rotation cannot contain only a single quaternion.

Derivation of the Rotation Operator: Using Two Quaternions

Say that we have two general quaternions from the set Q called q and r and we have one pure quaternion from the set Q_0 called p. There are six combinations of products that we can get from p, q, r:

$$pqr, prq, qrp, rqp, rpq, \text{and } qpr.$$

We know that:

- Set Q is closed under multiplication.
- A rotation operator cannot consist of only one general quaternion.
- Since q and r are not distinct and they are both general quaternions, they can be treated the same.

Given the first two conditions, we know that four of the combinations here cannot work (the ones with qr or rq). Of the two that work, qpr and rpq, we know they can be treated the same due to the third condition.

So we let $q = q_0 + \mathbf{q}, p = 0 + \mathbf{p}$, and $r = r_0 + \mathbf{r}$. The real part of the product qpr is:

$$-r_0(\mathbf{q} \cdot \mathbf{p}) - q_0(\mathbf{p} \cdot \mathbf{r}) + (\mathbf{q} \times \mathbf{r}) \cdot \mathbf{p}$$

Derivation of the Rotation Operator: Using Two Quaternions

The real part should be 0. We can do this by setting $r_0 = q_0$. Doing so results in:

$$-q_0(\mathbf{q} + \mathbf{r})\mathbf{p} + (\mathbf{q} \times \mathbf{r}) \cdot \mathbf{p}$$

This can become 0 if $r = -q$. Note that if we set $r = -q$, we also get the following:

$$r = r_0 + r = q_0 - q = q^* \rightarrow q = r^*$$

Even though there is no real distinction between the two possible combinations, we will still write both:

$$qpq^* \text{and } q^*pq$$

So when we have an input vector v, we have two possible triple-product quaternion operators defined as:

$$w_1 = qvq^*$$

$$w_2 = q^*vq$$

We can write these generally as a map $W_q : Q_0 \rightarrow Q_0$ where the function is defined as $W_q(v) = qvq^*$.

Proof

Proof that $W_q(v)$ is a Rotation

While we have a lot of evidence for $W_q(v)$ being a rotation operator, we have to prove it. To do so, we need two other theorems to help.

Theorem

W_q is a linear operator meaning that given any two vectors x and y in $\mathbb{R}^3$ and a scalar quantity k, we can say that:

$$W_q(kx + y) = kW_q(x) + W_q(y)$$

To prove that this is true, let us start by using the distributive property on the quaternion product:

$$W_q(kx + y) = q(kx + y)q^*$$

$$W_q(kx + y) = (kqx + qb)q^*$$

$$W_q(kx + y) = kqxq^* + qyq^*$$

$$W_q(kx + y) = kW_q(x) + W_q(y)$$

□

Proof that $W_q(v)$ is a Rotation

Theorem

When W_q is applied to a vector $v = k\mathbf{q}$, such that v is in the same direction as the vector component of q, the vector remains unchanged.

$$W_q(v) = qvq^*$$

$$W_q(v) = q(k\mathbf{q})q^*$$

$$W_q(v) = (2q_0^2 - 1)(k\mathbf{q}) + 2(\mathbf{q} \cdot k\mathbf{q})\mathbf{q} + 2q_0(\mathbf{q} \times k\mathbf{q})$$

$$W_q(v) = kq_0^2\mathbf{q} - k|\mathbf{q}|^2\mathbf{q} + 2k|\mathbf{q}|^2\mathbf{q}$$

$$W_q(v) = k(q_0^2 + |\mathbf{q}|^2)\mathbf{q}$$

$$W_q(v) = k\mathbf{q}$$

□

Proof that $W_q(v)$ is a Rotation

To start, we state:

$$q = q_0 + \mathbf{q} = \cos\theta + u\sin\theta$$

$$u = \frac{\mathbf{q}}{|\mathbf{q}|}$$

In this case, u is the unit vector.

We write v as $v = a + n$. The vector a is a scalar multiple of $\mathbf{q}$ because the vector a lies along the vector $\mathbf{q}$, so $a = k\mathbf{q}$ for some scalar quantity k. If we use the theorem on the previous slide, we can say that:

$$W_q(a) = W_q(k\mathbf{q}) = k\mathbf{q} = a$$

We now prove that W_q rotates the component n through an angle of 2θ about $\mathbf{q}$ as an axis. To do this, we compute the following using the fact that $u = \frac{\mathbf{q}}{|\mathbf{q}|}$:

$$W_q(n) = (q_0^2 - |\mathbf{q}|^2)n + 2(\mathbf{q}\cdot n)\mathbf{q} + 2q_0(\mathbf{q}\times n)$$

$$W_q(n) = (q_0^2 - |\mathbf{q}^2|)n + 2q_0(\mathbf{q}\times n)$$

$$W_q(n) = (q_0^2 - |\mathbf{q}^2|)n + 2q_0|\mathbf{q}|(u\times n)$$

Proof that $W_q(v)$ is a Rotation

Rewrite $u \times n$ as $n_\perp$. We rewrite the last equation on the previous slide as $W_q(n) = (q_0^2 - |\mathbf{q}|^2)n + 2q_0|\mathbf{q}|n_\perp$. Our goal now is to prove that n and $n_\perp$ have the same length. Since the angle between the two vectors is $\frac{\pi}{2}$ and $\sin(\frac{\pi}{2}) = 1$, we state:

$$|n_\perp| = |n \times u| = |n||u|\sin(\frac{\pi}{2}) = |n|$$

We rewrite $W_q(n)$ as:

$$(\cos^2\theta - \sin^2\theta)n + (2\cos\theta\sin\theta)n_\perp =$$

$$\cos(2\theta)n + \sin(2\theta)n_\perp$$

So, we have shown that $W_q(v) = W_q(a + n) = W_q(a) + W_q(n) = a + m$ such that $m = W_q(n) = \cos(2\theta)n + \sin(2\theta)n_\perp$. So, we have proved the following:

Theorem

For any unit quaternion $q = q_0 + \mathbf{q} = \cos\theta + u\sin\theta$ and for any given vector $v \in \mathbb{R}^3$, the operator $W_q = qvq^*$ on vector v is a rotation of the vector v through an angle of 2θ about $\mathbf{q}$ as the axis of rotation. □

Application

The Aerospace Sequence in Euler Angles Form

We will tackle the aerospace sequence in two ways: through Euler angles and through quaternions. Euler angles are the angle of rotation about a coordinate axis. A sequence of these rotations is called an Euler angle sequence or an Euler angle-axis sequence. Successive axes of rotations of an Euler angle sequence have to be distinct; given this restriction, there are 12 Euler angle-axis sequences.

$$
\begin{array}{ccc}
xyz & yzx & zxy \\
xzy & yxz & zyx \\
xyx & yzy & zxz \\
xzx & yxy & zyz
\end{array}
$$

Visuals of the Aerospace Sequence

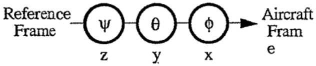

Figure: Aircraft Euler Angle Sequence

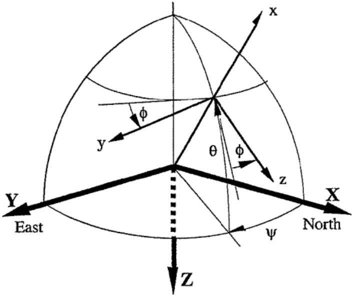

Figure: Geometric Representation of Aircraft Euler Angle Sequence

Aerospace Sequence Rotation Matrix: Euler Angles

The aerospace rotation sequence shown in the first figure in the previous slide can be represented mathematically as the following matrix product:

$$R = (R^x)_\phi (R^y)_\theta (R^z)_\psi$$

$$R = (R^x)_\phi \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ (\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi) & (\sin\psi\sin\theta\sin\phi + \cos\psi\cos\theta) & \cos\theta\sin\phi \\ (\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi) & (\sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi) & \cos\theta\cos\phi \end{bmatrix}$$

The Aerospace Sequence Applied

- The Euler angle-axis sequence zyx is important in aerospace.
- Seen in Heading and Attitude Indicator
- The positive x-axis of the aircraft is directed along the longitude and the positive y-axis is directed along the right wing, and the positive z-axis is normal to the x and y axes and it points downwards.
- Rotation through an angle ψ about the z-axis defines the aircraft's heading.
- Rotation about the y-axis through an angle θ defines the aircraft's elevation.
- Rotation about the x-axis through an angle of ϕ that defines the aircraft's bank angle.
- Put together, these three rotations connect the aircraft's body coordinate frame to the local reference coordinate frame of Earth.

Aerospace Sequence in Quaternion Form

To use the quaternion form, we state the following:

$$\alpha = \frac{\psi}{2}$$

$$\beta = \frac{\theta}{2}$$

$$\gamma = \frac{\phi}{2}$$

The half-angle equations above make it easier to do the rest of this process. Writing in terms of α, β, γ, the quaternions used to define the rotation operators are:

$$q_{z,\psi} = \cos\alpha + k\sin\alpha$$

$$q_{y,\theta} = \cos\beta + j\sin\beta$$

$$q_{x,\phi} = \cos\gamma + i\sin\gamma$$

Aerospace Sequence in Quaternion Form

Since this is a sequence of frame rotations, the quaternion product representing a composite rotation is $q = q_{z,\psi} q_{y,\theta} q_{x,\phi}$. When we calculate this composite product, we get:

$$q = q_{z,\psi} q_{y,\theta} q_{x,\phi} = q_0 + iq_1 + jq_2 + kq_3$$

The above equation holds true when:

$$q_0 = \cos\alpha\cos\beta\cos\gamma + \sin\alpha\sin\beta\sin\gamma$$

$$q_1 = \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma$$

$$q_2 = \cos\alpha\sin\beta\cos\gamma + \sin\alpha\cos\beta\sin\gamma$$

$$q_3 = \sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma$$

The rotation axis can be defined as:

$v = (v_1, v_2, v_3)$ such that the following conditions are true:

$$v_1 = q_1$$

$$v_2 = q_2$$

$$v_3 = q_3$$

Note

The paper that accompanies this slide presentation has more applications such as orbit and orbit ephemeris sequences, but due to the limited time of the presentation, only the aerospace sequence is mentioned.