Bonnet-Myers Theorem

Grace Howard

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Some History

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- In 1941, Sumner Byron Myers showed that only a lower bound on Ricci curvature was needed to come to the same conclusion.

Manifold

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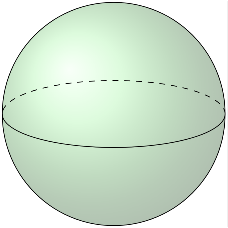
For example, the Earth's surface (approximately a sphere) is a $2-$ manifold because it locally resembles $2-$ dimensional Euclidean space.



Tangent Space

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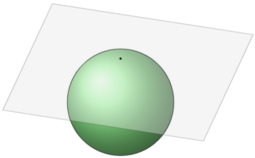

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  makes $\mathbb{R}^n$ into a Riemannian manifold. Every submanifold of $\mathbb{R}^n$ inherits a metric by restricting the Euclidean metric to M. S^{n-1} inherits a metric making it into a Riemannian manifold.

Ricci Curvature

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  - $ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$
  - $(g_{ij}) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} \qquad (g^{ij}) = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2\theta} \end{pmatrix}$
  - $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$
  - $R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$
  - $R_{\mu\nu} = \mathrm{Ric}\left(\frac{\partial}{\partial x_\mu}, \frac{\partial}{\partial x_\nu}\right) = R^{\lambda}_{\mu\lambda\nu}.$

Geodesic

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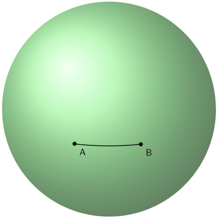

Complete Manifold

- A *complete* Riemannian manifold is one wherein each geodesic is isometric to the real line.
  - The n dimensional sphere is a compact $n-$manifold.
  - All compact Riemannian manifolds are geodesically complete.

Diameter

The diameter of a Riemannian manifold (M, g) is

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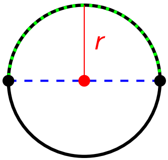

For example,

$$S^1(r) = \{x \in \mathbb{R}^2 : |x| = r\},$$

has

$$\mathrm{diam}(S^1(r), S^1) = \pi r \qquad \text{and} \qquad \mathrm{diam}(S^1(r), \mathbb{R}^2) = 2r.$$

Bonnet-Myers Theorem

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Theorem

Let M be complete Riemannian manifold of dimension n whose Ricci curvature satisfies

$$\mathrm{Ric}(u, u) \geq \frac{n-1}{r^2}$$

for all $u \in T_pM$ for all $p \in M$ with $r > 0$.Then,

$$\mathrm{diam}(M, g) \leq \pi r$$

and M is compact.

Proof Outline

- By the Hopf-Rinow theorems, any two points $p, q \in M$ can be joined by a length minimizing geodesic.

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- If every sufficiently long geodesic, satisfying say $\ell(\gamma) > L$ doesn't minimize length, then M is necessarily of diameter at most L.

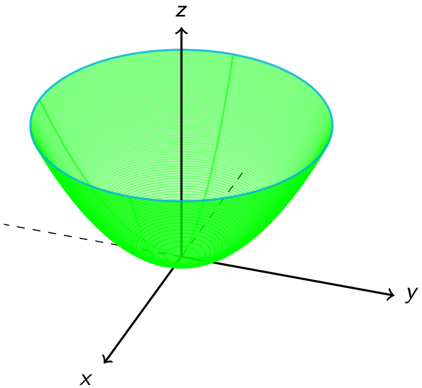
Bonnet-Myers Theorem

The condition on the Ricci curvature cannot be weakened to $\mathrm{Ric}(u, u) > 0$ for all unit vectors.

Condition

Consider an elliptic paraboloid of revolution given by

$$F(x, y) = x^2 + y^2.$$



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$$K = \frac{F_{xx}F_{yy} - F_{xy}^2}{(1 + F_x^2 + F_y^2)^2}.$$

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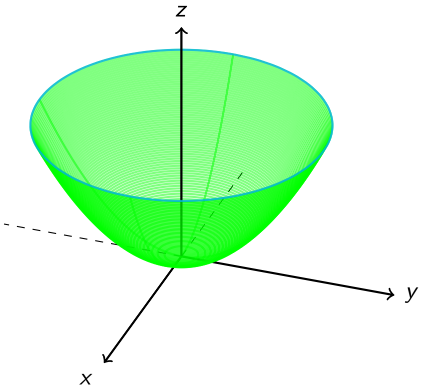

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$$F_x = 2x \quad F_y = 2y$$
$$F_{xx} = 2 \quad F_{yy} = 2 \quad F_{xy} = 0$$

$$K = \frac{4}{(1 + 4x^2 + 4y^2)^2}.$$

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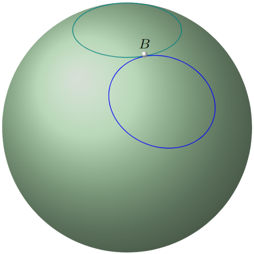
- If the sectional curvature satifies $K > 0$, then the Ricci curvature is greater than 0.
- Therefore, the curvature is positive.
- However, the manifold is not compact because it is unbounded.

Fundamental Group

This sphere has two loops passing through the point B.



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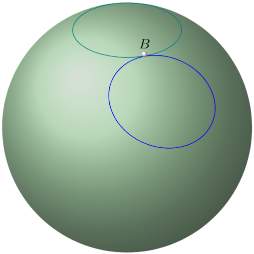

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Conclusion

Thank you for listening.