BONNET-MYERS THEOREM

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Abstract. Bonnet-Myers Theorem says that if a complete Riemannian manifold has positive curvature obeying some bound, then the manifold is compact. In other words, it relates local properties of the manifold to the topological properties. Additionally, it says that if the manifold is complete and has positive curvature which is bounded from below, then the fundamental group of the manifold is finite.

1. Introduction

Bonnet-Myers Theorem relates local properties of a Riemannian manifold to the topological properties of the manifold. A manifold is a topological space that is locally Euclidean. A Riemannian manifold is a smooth manifold, which broadly means that you can do calculus on the manifold, with a Riemannian metric. A geodesically complete manifold, or just a complete manifold, is a Riemannian manifold for which, starting at any point on the manifold, there are straight paths extending infinitely in all directions.

Bonnet-Myers Theorem states that if a Riemannian manifold M is complete and has positive, bounded-below curvature, then M is compact. In 1855, the theorem was proven for surfaces by Pierre Ossian Bonnet. In this special case, the notions of curvature (Gauss, sectional, and Ricci) are all the same.

In 1941, Sumner Byron Myers showed that only a lower bound on Ricci curvature was needed to come to the same conclusion.

More formally, the statement of the theorem is as follows:

Theorem 1.1 (Bonnet-Myers)**.** Let (M, g) be complete Riemannian manifold of dimension n whose Ricci curvature satisfies

$$\mathrm{Ric}(g) \geq \frac{n-1}{r^2}$$

for all $v \in SM = \{w \in TM : ||w|| = 1\}$.Then,

$$\mathrm{diam}(M, g) \leq \pi r.$$

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This can be used in the proof of a similar result:

Theorem 1.2 (Bonnet-Myers)**.** Let M be a compact Riemannian manifold with positive Ricci curvature; then its fundamental group is finite.

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3. Background

To begin, some relevant elementary definitions are recounted. Additionally, information can be found in [Krö10] and [Gud04].

Definition 3.1 (Einstein Summation Convention)**.** If an index variable appears twice in an expression, once as an upper index in one term and again as a lower index in another term, then the expression is a summation over all possible values of that index. For example,

$$a_i b^i = \sum_i a_i b^i.$$

Definition 3.2 (Inner product)**.** In a real vector space, an *inner product* $\langle ., . \rangle$ satisfies the following four properties. Let u, v, and w be vectors and let α be a scalar, then:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.
- $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$.
- $\langle v, w \rangle = \langle w, v \rangle$.
- $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Definition 3.3 (Inner product space)**.** An *inner product space* is a vector space V along with an inner product on V.

Definition 3.4 (Orthonormality)**.** Let V be an inner-product space. A set of vectors

$$\{u_1, u_2, \ldots, u_n, \ldots\} \in V$$

is called *orthonormal* if and only if for all i, j

$$\langle u_i, u_j \rangle = \delta_{ij}.$$

4. Manifolds

A manifold is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset of $\mathbb{R}^n$. For a more formal definition, there are some additional definitions which are helpful.

Definition 4.1 (Open ball)**.** An *open ball* is the set of all points $x \in \mathbb{R}^n$ such that $|x-y| < r$ for some fixed $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ where

$$|x-y| = \left[\sum_i \left(x^i - y^i\right)^2\right]^{\frac{1}{2}}.$$

Definition 4.2 (Open set)**.** A set $U \subset \mathbb{R}$ is open if for every $x \in U$ there exists $\epsilon > 0$ such that$(x-\epsilon, x+\epsilon) \subset U$. In other words, an *open set* in $\mathbb{R}^n$ is a set constructed from an arbitrary union of open balls. Or, $V \subset \mathbb{R}^n$ is open if, for any $y \in V$, there is an open ball centered at y that is completely inside V.

Definition 4.3 (Closed set)**.** A subset A of X is a *closed set* if and only if its complement, $A^c = X \backslash A$, is open.

Definition 4.4 (Topological space)**.** A topology on a nonempty set X is a collection of subsets of X such that:

- The empty set $\emptyset$ and the set X are open.
- The union of an arbitrary collection of open sets is open.
- The intersection of a finite number of open sets is open.

A collection $\mathcal{T}$ of subsets of X is a topology on X if:

- $\emptyset, X \in \mathcal{T}$.
- If $G_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} G_\alpha \in \mathcal{T}$.
- If $G_i \in \mathcal{T}$ for $i = 1, 2, \ldots, n$, the $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is a *topological space.*

Definition 4.5 (Topological manifold)**.** A topological space M is a *topological manifold* of dimension n if it has the following properties:

- M is a Hausdorff space: For every pair of points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.
- M is second countable: There exists a countable basis for the topology of M.
- M is locally Euclidean of dimension n: Every point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 4.6 (Smooth)**.** A map is C^∞, or *smooth*, if it is infinitely differentiable.

Definition 4.7 (Diffeomorphic)**.** Two sets M and N are *diffeomorphic* if there exists a C^∞ map $\phi : M \to N$ with a C^∞ inverse $\phi^{-1} : N \to M$. The map ϕ is called a diffeomorphism.

Definition 4.8 (Chart)**.** A chart consists of a subset U of a set M, along with an injective map $\phi : U \to \mathbb{R}^n$, such that the image $\phi(U)$ is open in $\mathbb{R}^n$.

Definition 4.9 (Atlas)**.** A smooth *atlas* is an index collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that satisfies:

- The union of the U_α cover M, that is $\bigcup_\alpha U_\alpha = M$.
- If two charts overlap, $U_\alpha \cap U_\beta \neq \emptyset$, then the map $(\phi_\alpha \circ \phi_\beta^{-1})$ takes all points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ onto an open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$, and all of these maps are C^∞ where they are defined.

Definition 4.10 (Manifold)**.** A C^∞ $n-$dimensional *manifold* is a set M along with a maximal atlas, one that contains every possible compatible chart.

With that, consider the following example of a manifold:

Example 4.1. The sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

is an $n-$manifold. Let

$$U_1 = S^n \backslash \{(0, \ldots, 0, 1)\}$$

and

$$U_2 = S^n \backslash \{(0, \ldots, 0, -1)\}.$$

Then, $U_1 \cup U_2 = S^n$. Let $\phi_1(x_1, x_2, \ldots, x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right)$. The map $\phi_1 : U_1 \to \mathbb{R}^n$ is called stereographic projection. The inverse map, $\phi_1^{-1} : \mathbb{R}^n \to U_1$ is defined by

$$\phi_1^{-1}(y_1, \ldots, y_n) = \left(\frac{2y_1}{\sum_{i=1}^n y_i^2 + 1}, \ldots, \frac{2y_n}{\sum_{i=1}^n y_i^2 + 1}, 1 - \frac{2}{\sum_{i=1}^n y_i^2 + 1}\right).$$

Both ϕ_1 and ϕ_1^{-1} are continuous, and thus ϕ_1 is a homeomorphism.

The second coordinate chart (U_2, ϕ_2), stereographic projection from the south pole, is given by $\phi_2 = -\phi_1 \circ (-1)$, where (-1) is multiplication by -1 on the sphere. Since multiplication by -1 is a homeomorphism of the sphere to itself, the map $\phi_2 : U_2 \to \mathbb{R}^n$ is a homeomorphism. Next,

$$\phi_2 \circ \phi_1^{-1}(y_1, \ldots, y_n) = \frac{1}{\sum_{i=1}^n y_i^2}(y_1, \ldots, y_n)$$

and $\phi_2 \circ \phi_1^{-1} = \phi_1 \circ \phi_2^{-1}$. Hence, S^n is an $n-$ manifold.

5. Riemannian Manifolds

A Riemannian manifold is a smooth manifold equipped with a Riemannian metric. This is a choice at each point on the manifold of a positive definite inner product on the tangent space at the point.

Definition 5.1 (Equivalent)**.** Given a C^k $n-$ dimensional manifold M, for any $p \in M$, two C^1 curves, $\gamma_1 : [-\epsilon_1, \epsilon_1] \to M$ and $\gamma_2 : [-\epsilon_2, \epsilon_2] \to M$, through p, meaning $\gamma_1(0) = \gamma_2(0) = p$, are *equivalent* if and only if there is some chart (U, φ) at p so that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Definition 5.2 (Tangent vector)**.** Given any C^k $n-$ dimensional manifold M, with $k \geq 1$, for any $p \in M$, a *tangent vector* to M at p is an equivalence class of C^1 curves through p on M, modulo the equivalence relation defined previously. The set of all tangent vectors at p is denoted $T_p(M)$.

Note that $T_p(M)$ is a vector space of dimension n, the dimension of the manifold.

Definition 5.3 (Riemannian metric)**.** A *Riemannian metric* on a smooth manifold M is a choice at each point $x \in M$ of a positive definite inner product $\langle , \rangle$ on T_xM, the inner products varying smoothly with x .

Example 5.1. Consider the parametrization of the sphere S^2 in terms of angles θ and φ as follows:

$$x = \sin\theta\cos\varphi$$

$$y = \sin\theta\sin\varphi$$

$$z = \cos\theta.$$

Restrict the domain to

$$V = \{(\theta, \varphi) : 0 < \theta < \pi, 0 < \varphi < 2\pi\}.$$

To compute the matrix which gives a Riemannian metric, a basis $(u(\theta, \varphi), v(\theta, \varphi))$ of the tangent plane T_pS^2 at $p = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ must be found. For this, use

$$u(\theta, \varphi) = \frac{\partial p}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi. -\sin\theta)$$
$$v(\theta, \varphi) = \frac{\partial p}{\partial \varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0).$$

From this,

$$\langle u(\theta,\varphi), u(\theta,\varphi)\rangle = 1$$
$$\langle u(\theta,\varphi), v(\theta,\varphi)\rangle = 0$$
$$\langle v(\theta,\varphi), v(\theta,\varphi)\rangle = \sin^2\theta.$$

From this, the metric on T_pS^2 with respect to the basis is given by

$$g_p = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

Thus, for any tangent vector w,

$$g_p(w,w) = d\theta^2 + \sin^2\theta d\varphi^2.$$

Definition 5.4 (Riemannian manifold)**.** The smooth manifold M aforementioned is known as a *Riemannian manifold*.

Definition 5.5 (Pseudo Riemannian manifold)**.** A *pseudo-Riemannian manifold* (M, g) is a differentiable manifold M equipped with an everywhere non-degenerate, smooth, symmetric metric tensor g.

6. Differential Geometry

Definition 6.1 (Tensor)**.** An nth rank *tensor* in m dimensional space is an object that has n indices and m^n components and obeys certain transformation rules:

- $S = S'$, a scalar, which is a tensor of rank 0, is invariant under transformations.
- For a contravariant vector, or a tensor of rank 1,
$$V^\alpha = V^{\alpha'}\frac{\partial x^\alpha}{\partial x^{\alpha'}}.$$
- For a covariant vector, which is a tensor of rank 1,
$$V_\alpha = V_{\alpha'}\frac{\partial x^{\alpha'}}{\partial x^\alpha}.$$
- For a tensor of higher rank with mixed indices,
$$T^{\alpha\ldots}_{\beta\ldots} = T^{\alpha'\ldots}_{\beta'\ldots}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\beta'}}{\partial x^\beta}.$$
- Contraction, which is a summation over a pair of one covariant and one contravariant indices, creates a tensor of a lesser rank.

Definition 6.2 (Christoffel symbols)**.** The *Christoffel symbol* is defined as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right).$$

Definition 6.3 (Riemann Tensor)**.** The *Riemann tensor* is defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\gamma}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

and

$$R_{\rho\sigma\mu\nu} = g_{\lambda\rho} R^{\lambda}_{\sigma\mu\nu}.$$

Definition 6.4 (Ricci tensor)**.** Let M be a Riemannian manifold with curvature tensor R. For $p \in M$, define a linear map $T_p(M) \to T_p(M)$,

$$X \to -R(X, Y)Z$$

that depends on $Y, Z \in T_p(M)$. The trace of this linear map is defined to be the *Ricci tensor* $\rho(Y, Z)$. The *Ricci tensor* is also defined as

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}.$$

Definition 6.5 (Metric tensor)**.** A *metric tensor* at a point p of M is a bilinear form defined on the tangent space at p.

Definition 6.6 (Connection)**.** A *connection* on M associates to vector fields X, Y on M another vector field $\nabla_X Y$, called the covariant derivative of Y with repsect to X, satisfying the following conditions:

- ∇ is linear in X over the smooth functions on M, i.e. $\nabla_{fX}(Y) = f\nabla_X Y$.
- ∇ is a derivation in Y, meaning for $f : M \to \mathbb{R}$ smooth, $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

The following is an example of explicitly calculating the aforementioned values (for more, see [Car04]):

Example 6.1. For example, a metric for the three-sphere in coordinates (ψ, θ, ϕ) is

$$ds^2 = d\psi^2 + \sin^2\psi d\theta^2 + \sin^2\psi \sin^2\theta d\phi^2.$$

Since

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}),$$

the nonzero metric elements are

$$g_{\psi\psi} = 1 \qquad g_{\theta\theta} = \sin^2\psi \qquad g_{\phi\phi} = \sin^2\psi \sin^2\theta.$$

Consider firstly

$$\Gamma^{\psi}_{\mu\nu} = \frac{1}{2} g^{\psi\rho}(\partial_\mu g_{\nu\psi} + \partial_\nu g_{\psi\mu} - \partial_\psi g_{\mu\nu}).$$

Since the derivatives of $g_{\psi\psi}$ vanish, the first two terms are zero. Trying both choices for the last term, $\mu = \nu = \theta$ and $\mu = \nu = \phi$,

$$\Gamma^{\psi}_{\theta\theta} = -\sin\psi \cos\psi$$

$$\Gamma^{\psi}_{\phi\phi} = -\sin\psi\cos\psi\sin^2\theta.$$

All other symbols with upper ψ index are zero. Next,

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2}g^{\theta\theta}(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\theta\mu} - \partial_\theta g_{\mu\nu}).$$

The last term is nonzero only if $\mu = \nu = \phi$. From this,

$$\Gamma^{\theta}_{\phi\phi} = \sin\theta\cos\theta.$$

The only other way this is nonzero is when $\mu = \theta$ or $\nu = \theta$. From this,

$$\Gamma^{\theta}_{\theta\nu} = \frac{1}{2}g^{\theta\theta}(\partial_\theta g_{\nu\theta} + \partial_\nu g_{\theta\theta} - \partial_\theta g_{\theta\nu}).$$

By inspection, the first and final term are 0 by inspection. The middle term is nonzero when $\nu = \psi$. So,

$$\Gamma^{\theta}_{\theta\psi} = \Gamma^{\theta}_{\psi\theta} = \cot\psi.$$

Lastly,

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2}g^{\phi\phi}(\partial_\mu g_{\nu\phi} + \partial_\nu g_{\phi\mu} - \partial_\phi g_{\mu\nu}).$$

By examination, the last term will always be zero. For the symbol to be nonzero, it must be the case that either $\mu = \phi$ or $\nu = \phi$. So,

$$\Gamma^{\phi}_{\psi\phi} = \Gamma^{\phi}_{\phi\psi} = \cot\psi$$

and

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

So, the nonzero Christoffel symbols are

$$\Gamma^{\psi}_{\theta\theta} = -\sin\psi\cos\psi \qquad \Gamma^{\psi}_{\phi\phi} = -\sin\psi\cos\psi\sin^2\theta$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^{\theta}_{\theta\psi} = \Gamma^{\theta}_{\psi\theta} = \cot\psi$$

$$\Gamma^{\phi}_{\psi\phi} = \Gamma^{\phi}_{\phi\psi} = \cot\psi \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

From here, the Riemann tensor was defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}.$$

Recall that

$$R_{\rho\sigma\mu\nu} = g_{\lambda\rho}R^{\lambda}_{\sigma\mu\nu},$$

so

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \qquad R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \qquad R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \qquad R_{\rho[\sigma\mu\nu]} = 0$$

can be used. Firstly, since $g_{\psi\psi} = 1$, there is no conversion between the forms with upper index ψ. With that,

$$R_{\psi\sigma\mu\nu} = \partial_\mu\Gamma^{\psi}_{\nu\sigma} + \partial_\nu\Gamma^{\psi}_{\mu\sigma} + \Gamma^{\psi}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\psi}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}.$$

The first way to keep the first term nonzero is to set $\nu = \sigma = \theta$ and $\mu = \psi$ to get

$$\begin{aligned} R_{\psi\theta\psi\theta} &= \partial_\psi(-\sin\psi\cos\psi) - \partial_\theta \Gamma^\psi_{\psi\theta} + \Gamma^\psi_{\psi\lambda}\Gamma^\lambda_{\theta\theta} - \Gamma^\psi_{\theta\lambda}\Gamma^\lambda_{\psi\theta} \\ &= \sin^2\psi - \cos^2\psi + \cos^2\psi \\ &= \sin^2\psi. \end{aligned}$$

By the symmetries,

$$R_{\psi\theta\psi\theta} = \sin^2\psi \qquad R_{\theta\psi\psi\theta} = R_{\psi\theta\theta\psi} = -\sin^2\psi \qquad R_{\psi\psi\theta\theta} = 0.$$

Secondly, choose $\nu = \sigma = \phi$ and $\mu = \psi$ for the first term to be nonzero. Then,

$$\begin{aligned} R_{\psi\phi\psi\phi} &= \partial_\psi(-\sin\psi\cos\psi\sin^2\theta) - \partial_\phi \Gamma^\psi_{\psi\phi} + \Gamma^\psi_{\psi\lambda}\Gamma^\lambda_{\phi\phi} - \Lambda^\psi_{\phi\lambda}\Gamma^\lambda_{\psi\phi} \\ &= \sin^2\theta(\sin^2\psi - \cos^2\psi) + \cos^2\psi\sin^2\theta \\ &= \sin^2\theta\sin^2\psi. \end{aligned}$$

Then, the symmetries give

$$R_{\psi\phi\psi\phi} = \sin^2\theta\sin^2\psi \qquad R_{\phi\psi\psi\phi} = R_{\psi\phi\phi\psi} = -\sin^2\theta\sin^2\psi \qquad R_{\psi\psi\phi\phi} = 0.$$

Next, consider when $\nu = \sigma = \phi$ and $\mu = \theta$ to keep the first term nonzero, giving

$$R_{\psi\phi\theta\phi} = \partial_\theta(-\sin\psi\cos\psi\sin^2\theta) - \partial_\phi \Gamma^\psi_{\theta\phi} + \Gamma^\psi_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\psi_{\phi\lambda}\Gamma^\lambda_{\theta\phi}$$

$$= -2\sin\psi\cos\psi\sin\theta\cos\theta + \sin\psi\cos\psi\sin\theta\cos\theta + \sin\psi\cos\psi\sin\theta\cos\theta$$

$$= 0.$$

Every symbol with a nonzero second term has already been calculated with the symmetries. Similarly, the third term,

$$\Gamma^\psi_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} = \Gamma^\psi_{\mu\sigma}\Gamma^\theta_{\nu\sigma} + \Gamma^\psi_{\mu\phi}\Gamma^\phi_{\nu\sigma},$$

has already been calculated via symmetries. The last term is similar. So, the nonzero elements are

$$\begin{aligned} R_{\psi\theta\psi\theta} &= R_{\theta\psi\theta\psi} = \sin^2\psi & R_{\theta\psi\psi\theta} &= R_{\psi\theta\theta\psi} = -\sin^2\psi \\ R_{\psi\phi\psi\phi} &= R_{\phi\psi\phi\psi} = \sin^2\theta\sin^2\psi & R_{\phi\psi\psi\phi} &= R_{\psi\phi\phi\psi} = -\sin^2\theta\sin^2\psi \\ R_{\theta\phi\theta\phi} &= R_{\phi\theta\phi\theta} = \sin^2\theta\sin^4\psi & R_{\phi\theta\theta\phi} &= R_{\theta\phi\phi\theta} = -\sin^2\theta\sin^4\psi. \end{aligned}$$

Using the transformation stated above, where

$$g^{\psi\psi} = 1 \qquad g^{\theta\theta} = \frac{1}{\sin^2\psi} \qquad g^{\phi\phi} = \frac{1}{\sin^2\psi\sin^2\theta}.$$

So the final Riemann tensor elements are

$$R^\psi_{\theta\psi\theta} = \sin^2\psi \qquad R^\psi_{\theta\theta\psi} = -\sin^2\psi \qquad R^\psi_{\phi\psi\phi} = \sin^2\theta\sin^2\psi \qquad R^\psi_{\phi\phi\psi} = -\sin^2\theta\sin^2\psi$$

$$R^\theta_{\psi\theta\psi} = 1 \qquad R^\theta_{\psi\psi\theta} = -1 \qquad R^\theta_{\phi\theta\phi} = \sin^2\theta\sin^2\psi \qquad R^\theta_{\phi\phi\theta} = -\sin^2\theta\sin^2\psi$$

$$R^{\phi}_{\psi\phi\psi} = 1 \qquad R^{\phi}_{\psi\psi\phi} == -1 \qquad R^{\phi}_{\theta\phi\theta} = \sin^2\psi \qquad R^{\phi}_{\theta\theta\phi} = -\sin^2\psi.$$

So, for the Ricci tensor, $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$,

$$\begin{aligned}
R_{\psi\psi} &= R^{\psi}_{\psi\psi\psi} + R^{\theta}_{\psi\theta\psi} + R^{\phi}_{\psi\phi\psi} \\
&= 0 + 1 + 1 = 2, \\
R_{\theta\theta} &= R^{\psi}_{\theta\psi\theta} + R^{\theta}_{\theta\theta\theta} + R^{\phi}_{\theta\phi\theta} \\
&= \sin^2\psi + 0 + \sin^2\psi = 2\sin^2\psi, \\
R_{\phi\phi} &= R^{\psi}_{\phi\psi\phi} + R^{\theta}_{\phi\theta\phi} + R^{\phi}_{\phi\phi\phi} \\
&= \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta, \\
R_{\psi\theta} &= R^{\psi}_{\psi\psi\theta} + R^{\theta}_{\psi\theta\theta} + R^{\phi}_{\psi\phi\theta} \\
&= 0 + 0 + 0 = 0, \\
R_{\psi\phi} &= R^{\psi}_{\psi\psi\phi} + R^{\theta}_{\psi\theta\phi} + R^{\phi}_{\psi\phi\phi} \\
&= 0 + 0 + 0 = 0,
\end{aligned}$$

and

$$\begin{aligned}
R_{\theta\phi} &= R^{\psi}_{\theta\psi\phi} + R^{\theta}_{\theta\theta\phi} + R^{\phi}_{\theta\phi\phi} \\
&= 0 + 0 + 0 = 0.
\end{aligned}$$

With that, the independent components of the Ricci tensor are,

$$\begin{aligned}
&R_{\psi\psi} = 1 \\
&R_{\theta\theta} = 2\sin^2\psi \\
&R_{\phi\phi} = 2\sin^2\psi\sin^2\theta \\
&R_{\psi\theta} = R_{\psi\phi} = R_{\theta\phi} = 0.
\end{aligned}$$

Finally, the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$, is

$$R = g^{\psi\psi}R_{\psi\psi} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 2 + 2 + 2 = 6.$$

Definition 6.7 (Affine connection)**.** Let M be a differentiable manifold and $\mathfrak{X}(M)$ the set of differentiable vector fields on M. Let $X, Y, Z \in \mathfrak{X}(M)$ and f, g be differentiable real-valued functions on M. Then $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is an *affine connection* if it satisfies the following properties:

- $\nabla(fX + gY, Z) = f\nabla(X, Z) + g\nabla(Y, Z)$.
- $\nabla(X, Y + Z) = \nabla(X, Y) + \nabla(Y, Z)$.
- $\nabla(X, fY) = f\nabla(X, Y) + (Xf)Y$.

Definition 6.8 (Lie bracket)**.** The *Lie bracket* can be computed as

$$[X,Y] := \sum_{i=1}^{n}(X(Y^i) - Y(X^i))\partial_i = \sum_{i=1}^{n}\sum_{j=1}^{n}(X^j\partial_j Y^i - Y^i\partial_j X^i)\partial_i.$$

Definition 6.9 (Torsion tensor)**.** Let M be a manifold with an affine connection on the tangent bundle ∇. The *torsion tensor* of ∇ is the vector-valued $2-$form defined on vector fields X and Y by

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

where $[X,Y]$ is the Lie bracket of two vector fields.

Definition 6.10 (Curvature tensor)**.** Given a connection ∇ on the manifold M, define the *curvature tensor* R by

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Definition 6.11 (Levi-Civita connection)**.** An affine connection ∇ is called a *Levi-Civita connection* if:

- It preserves the metric, $\nabla g = 0$.
- It is torsion free.

Theorem 6.1 (Levi-Civita)**.** Every pseudo Riemannian manifold (M,g) has a unique Levi-Civita connection ∇.

Proof. Note that a metric g is compatible with ∇ if and only if

$$\nabla_{Xg} = 0$$

for all X. Since covariant differentiation commutes with contractions,

$$X(g(x_1,X_2)) = (\nabla_X X_1, X_2) + g(X_1, \nabla_X X_2)$$

for all X, X_1, X_2. Suppose that $M \subset \mathbb{R}^n$ and there is a connection associated to g. Consider that

$$\partial_k g(\partial_i,\partial_j) = g(\nabla_{\partial_k}\partial_i,\partial_j) + g(\partial_i,\nabla_{\partial_k}\partial_j).$$

Then, by cyclic permutation,

$$\partial_i g(\partial_j,\partial_k) = g(\nabla_{\partial_i}\partial_j,\partial_k) + g(\partial_j,\nabla_{\partial_i}\partial_k)$$

and

$$\partial_j g(\partial_k,\partial_i) = g(\nabla_{\partial_j}\partial_k,\partial_i) + g(\partial_k,\nabla_{\partial_j}\partial_i).$$

Define $S_{ij} = \nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ and $T_{ijk} = \partial_i g(\partial_j,\partial_k)$. These equations can be written

$$T_{kij} = g(S_{ik},\partial_j) + g(S_{jk},\partial_i),$$
$$T_{ijk} = g(S_{ij},\partial_k) + g(S_{ik},\partial_j),$$

and

$$T_{jki} = g(S_{jk},\partial_i) + g(S_{ij},\partial_k).$$

The unknowns in this equation are

$$g(S_{ik}, \partial_j) \qquad g(S_{jk}, \partial_i) \qquad g(S_{ij}, \partial_k).$$

Since the system is nonsingular, there is a unique solution and unique S_{ij}'s

Next, to show existence, choose S_{ij} to satisfy the system if three equations where $i < j < k$. Set $S_{ji} = S_{ij}$. We have a connection ∇ with $\nabla_{\partial_i \partial_j} = S_{ij}$ since the vector fields ∂_i are a basis at each tangent space on M. It is symmetric, since the torsion T vanishes on pairs (∂_i, ∂_j), and hence identically-since it is a tensor.

Note that the difference between the two terms in

$$X(g(X_1, X_2)) = (\nabla_{Xg})(X_1, X_2) + g(\nabla_X X_1, X_2) + g(X_1, \nabla_X X_2)$$

vanishes when X, X_1, X_2 are of the form ∂_i. Compatibility follows from this, as the difference of the two sides,

$$(\nabla_{Xg})(X_1, X_2)$$

is a tensor.

Definition 6.12 (Contravariant tensor)**.** A *contravariant tensor* is a tensor which satisfies specific transformation properties. Consider a vector

$$dr = dx_1 \hat{x_1} + dx_2 \hat{x_2} + dx_3 \hat{x_3}$$

where

$$dx'_i = \frac{\partial x'_i}{\partial x_j} dx_j.$$

Let $A_i := dx_i$. Then, any set of quantities A_j which transform in accordance with

$$A'_i = \frac{\partial x'_i}{\partial x_j} A_j$$

is a contravariant tensor.

Definition 6.13 (Covariant tensor)**.** To turn a *contravariant tensor* a^ν to a covariant tensor a_μ, use the metric tensor $g_{\mu\nu}$ to write

$$g_{\mu\nu} a^\nu = a_\mu.$$

Definition 6.14 (Comma derivative)**.** The components of the gradient of the one-form dA are denoted $A_, k$ and are given by

$$A_{,k} = \frac{\partial A}{\partial x^k}.$$

Definition 6.15 (Covariant derivative)**.** The *covariant derivative* of contravariant tensor A^a is given by

$$A^a_{;b} = \frac{\partial A^a}{\partial x^b} + \Gamma^a_{bk} A^k = A^a_{,b} + \Gamma^a_{bk} A^k.$$

The covariant derivative of a covariant tensor A_a is

$$A_{a;b} = \frac{\partial A_a}{\partial x^b} - \Gamma^k_{ab} A_k.$$

Definition 6.16 (Geodesic)**.** A curve $\gamma(t)$ on a surface S is called a *geodesic* if at every point $\gamma''(t)$ is either zero or parallel to its unit normal $\mathbf{n}$.

Definition 6.17 (Exponential map)**.** Let $v \in T_pM$ be a tangent vector to the manifold at p. Then there is a unique geodesic $\gamma_v : [0,1] \to M$ satisfying $\gamma_v(0) = p$ with initial tangent vector $\gamma'_v(0) = v$. The corresponding *exponential map* is defined by $\mathbf{exp}_p(v) = \gamma_v(1)$.

Definition 6.18 (Geodesically complete)**.** M is *geodesically complete* if $\mathbf{exp}$ can be defined on all of TM.

Theorem 6.2 (Hopf-Rinow 1)**.** The following are equivalent:

- M is geodesically complete.
- In the metric d on M induced by g, M is a complete metric space.

Proof. Suppose the second item to be true. Let M be complete in the appropriate metric. Then, if $\gamma : I \to M$ for I and open interval (a,b) is a geodesic, consider a sequence $b_n \to b$. Then,

$$d(\gamma(b_n), \gamma(b_m)) \leq \ell(\gamma_{[b_n,b_m]}) = O(|b_n - b_m|),$$

since geodesics move at constant speed. From this, the $\gamma(b_n)$ form a Cauchy sequence, converging to some point $p \in M$. In local coordinates, we can write $\gamma = (\gamma_1, \ldots, \gamma_n)$, when the geodesic property implies

$$\dddot{\gamma}_i = -\sum_{j,k} \Gamma^i_{jk} \dot{\gamma}_j \dot{\gamma}_k$$

for suitable Christoffel symbols Γ^i_{jk}.

Since the first derivatives $\dot{\gamma}_j$ are bounded, the second derivatives are too. Therefore, γ extends to the interval $(a,b]$ with a right-handed derivative $\dot{\gamma}$ at b. Furthermore, the right-handed derivative at b exists and is uniformly continuous in a neighborhood of b by the mean value theorem (ref). Now there is locally a geodesic $\gamma_1 : (b - \epsilon, b + \epsilon)$ at p with $\dot{\gamma}_1(b) - \dot{\gamma}(b)$. The function

$$\gamma_2(t) = \begin{cases} \gamma(t) & \text{if } t \leq b \\ \gamma_1(t) & \text{otherwise} \end{cases}$$

satisfies the geodesic equation everywhere and is defined on $(a, b + \epsilon)$. So, these geodesics can be extended to the right. The left is handled in a similar way.

Theorem 6.3 (Hopf-Rinow 2)**.** Suppose $\mathbf{exp}_p$ is defined on all of $T_p(M)$. Then for any $q \in M$, there is a geodesic γ from p to q that minimizes length.

Proof. Consider a small sphere

$$S_r(p) = \{x \in \mathbb{R}^n : |x - p| < r\}$$

with respect to the metric d. Take the point $p' \in S_r(p)$ with $d(p', q)$ minimized. From this,

$$d(p, q) = d(p', q) + r.$$

There is a geodesic γ travelling at unit speed with $\gamma(0)p$, $\gamma(r) = p'$. In particular,

$$d(\gamma(r), q) = d(p, q) - r.$$

Let S be the set of all s with

$$d(\gamma(s), q) = d(p, q) - s.$$

means that $r \in S$ and S is closed. If $d(p, q) \in S$, then we have a geodesic from p to q that minimizes length. Since $S \cap [0, d(p, q)]$ is closed, pick its largest element $s \in S \cap [0, d(p, q))$, and let $u = \gamma(s)$. Choose a small neighborhood $D_{2\delta}(u)$. From this, if we pick the point $u' \in S_\delta(u)$ closest to q, we have

$$d(u', q) = d(u, q) - \delta = d(p, q) - s - \delta.$$

Pick $u' = \gamma(s + \delta)$. First,

$$d(p, u') \geq d(q, p) - d(q, u') = d(p, q) - s + \delta.$$

The path γ from p to u connected with the geodesic from u to u' forms a path from p to u' of minimizing length

$$d(p, q) + \delta = d(p, q) - s + \delta,$$

so it is smooth and a geodesic. In particular, $s + \delta \in S$, which is a contradiction.

Definition 6.19 (Energy)**.** Let $c : I \to M$ be a smooth path in M. The *energy* is defined as

$$E(c) := \frac{1}{2}\int g(c', c').$$

Theorem 6.4 (First Variation Formula)**.**

- For any variation H of γ, we have

$$\frac{d}{ds}E(\gamma_s)|_{s=0} = g(Y(t), \dot\gamma(t))|_0^T - \int_0^T g\left(Y(t), \frac{\nabla}{dt}\dot\gamma(t)\right) dt.$$

- The critical points for all variations H of γ are geodesics.
- If $|\dot\gamma_s(t)|$ is constant for each fixed $s \in (-\epsilon, \epsilon)$, and $|\dot\gamma(t)| = 1$, then

$$\frac{d}{ds}E(\gamma_s)|_{s=0} = \frac{d}{ds}\ell(\gamma)_s)|_{s=0}.$$

- If γ is a critical point of the length, then it must be a reparametrization of a geodesic.

Proof.

- We have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}g(\dot\gamma_s(t), \dot\gamma_s(t)) &= g\left(\frac{\nabla}{ds}\dot\gamma_s(t), \dot\gamma_s(t)\right)\\ &= g\left(\frac{\nabla}{ds}\frac{\partial H}{\partial s}(t,s), \frac{\partial H}{\partial t}(t,s)\right)\\ &= \frac{\partial}{\partial t}g\left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}\right) - g\left(\frac{\partial H}{\partial s}, \frac{\nabla}{dt}\frac{\partial H}{\partial t}\right).\end{aligned}$$

  Then, by integrating from 0 to T with respect to t with $s = 0$ gives the desired result since

$$\frac{\partial H}{\partial s}|_{s=0} = Y \qquad \text{and} \qquad \frac{\partial H}{\partial t}|_{s=0} = \dot\gamma.$$

- If γ is a geodesic, then

$$\frac{\nabla}{dt}\dot\gamma(t) = 0.$$

  From this, the integral on the right-hand side vanishes. Similarly, since $Y(0) = 0 = Y(T)$, the rand-hand side vanishes. Conversely, if γ is a critical point for E, choose H with

$$Y(t) = f(t)\frac{\nabla}{dt}\dot\gamma(t)$$

for some $f \in C^\infty[0,T]$ such that $0 = f(T) = f(0)$. Then, it is known that

$$\int_0^T f(t) \left| \frac{\nabla}{dt}\dot\gamma(t) \right|^2 dt = 0$$

is true for all f. From this,

$$\frac{\nabla}{dt}\dot\gamma = 0.$$

- Fix $[0,T]$. Then, for all H, it is the case that

$$E(\gamma_s) = \frac{\ell(\gamma_s)^2}{2T}.$$

From this,

$$\frac{d}{ds}E(\gamma_s)|_{s=0} = \frac{1}{T}\ell(\gamma_s)\frac{d}{ds}\ell(\gamma_s)|_{s=0}.$$

When $s = 0$, the curve is parameterized by arc-length, so $\ell(\gamma_s) = T$.
- Using reparametrization, it can be assumed that $|\dot\gamma| = 1$. Then, γ is a critical point for ℓ, hence for E, and is therefore a geodesic.

Theorem 6.5 (Second Variation Formula)**.** Let $\gamma(t) : [0,T] \to M$ be a geodesic with $|\dot\gamma| = 1$. Let $H(t,s)$ be a variation of γ. Let

$$Y(t,s) = \frac{\partial H}{\partial s}(t,s) = (dH)_{(t,s)}\frac{\partial}{\partial s}.$$

- We have

$$\frac{d^2}{ds}E(\gamma_s)|_{s=0} = g\left(\frac{\nabla Y}{ds}(t,0), \dot\gamma\right)|_0^T + \int_0^T (|Y'|^2 - R(Y,\dot\gamma,Y\dot\gamma))dt.$$

- We have

$$\frac{d^2}{ds^2}\ell(\gamma_s)|_{s=0} = g\left(\frac{\nabla Y}{ds}(t,0), \dot\gamma(t)\right)|_0^T$$

$$+\int_0^T (|Y'|^2 - R(Y,\dot\gamma,Y,\dot\gamma) - g(\dot\gamma,Y')^2)dt,$$

where R is the $(4,0)$ curvature tensor, and

$$Y'(t) = \frac{\nabla Y}{dt}(t,0).$$

Letting

$$Y_n = Y - g(Y,\dot\gamma)\dot\gamma$$

for the normal component of Y, we can write this as

$$\frac{d^2}{ds^2}\ell(\gamma_s)|_{s=0} = g\left(\frac{\nabla Y_n}{ds}(t,0), \dot\gamma(t)\right)|_0^T + \int_0^T (|Y_n'|^2 - R(Y_n, \dot\gamma, Y_n, \dot\gamma))dt.$$

Proof. Use

$$\frac{d}{ds}E(\gamma_s) = g(Y(t,s), \dot\gamma_s(t))|_{t=0}^T - \int_0^T g\left(Y(t,s), \frac{\nabla}{dt}\dot\gamma_s(t)\right) dt.$$

Taking the derivative (with respect to s) gives

$$\frac{d^2}{ds^2}E(\gamma_s) = g\left(\frac{\nabla Y}{ds}, \dot\gamma\right)|_{t=0}^T + g\left(Y, \frac{\nabla}{ds}\dot\gamma)s\right)|_{t=0}^T$$

$$-\int_0^T \left(g\left(\frac{\nabla Y}{ds}, \frac{\nabla}{dt}\dot\gamma_s\right) + g\left(Y, \frac{\nabla}{ds}\frac{\nabla}{dt}\dot\gamma\right)\right) dt.$$

From this, use that

$$\frac{\nabla}{ds}\frac{\nabla}{dt}\dot\gamma_s(t) = \frac{\nabla}{dt}\frac{\nabla}{ds}\dot\gamma_s(t) + R\left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}\right)\dot\gamma_s$$

$$= \left(\frac{\nabla}{dt}\right)^2 Y(t,s) + R\left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial t}\right)\dot\gamma_s.$$

Now, setting $s = 0$, this gives

$$\frac{d^2}{ds^2}E(\gamma_s) = g\left(\frac{\nabla Y}{ds}, \dot\gamma\right)|_0^T + g\left(Y, \frac{\nabla\dot\gamma}{ds}\right)|_0^T$$

$$-\int_0^T \left[g\left(Y, \left(\frac{\nabla}{dt}\right)^2 Y\right) + R(\dot\gamma, Y, \dot\gamma, Y)\right] dt.$$

Next, using integration by parts,

$$-\int_0^T g\left(Y, \left(\frac{\nabla}{dt}\right)^2 Y\right) dt = -g\left(Y, \frac{\nabla}{dt}Y\right)|_0^T + \int_0^T \left|\frac{\nabla Y}{dt}\right|^2 dt.$$

Noting that

$$\frac{\nabla}{ds}\dot\gamma(s) = \frac{\nabla}{dt}Y(t,s),$$

$$\frac{d^2}{ds^2}E(\gamma_s)|_{s=0} = g\left(\frac{\nabla Y}{ds}, \dot\gamma\right)|_0^T + \int_0^T (|Y'|^2 - R(Y, \dot\gamma, Y, \dot\gamma))dt.$$

Differentiating to prove the variation of length,

$$\frac{d}{ds}\ell(\gamma_s) = \int_0^T \frac{1}{2\sqrt{g(\dot\gamma_s, \dot\gamma_s)}}\frac{\partial}{\partial s}g(\dot\gamma_s, \dot\gamma_s)dt.$$

Taking the derivative again gives

$$\frac{d^2}{ds^2}\ell(\gamma_s)|_{s=0} = \int_0^T \left[\frac{1}{2}\frac{\partial^2}{\partial s^2} g(\dot\gamma_s, \dot\gamma_s)|_{s=0} - \frac{1}{4}\left(\frac{\partial}{\partial s} g(\dot\gamma_s, \dot\gamma_s)\right)^2\right] dt,$$

where it is important to note that $g(\dot\gamma, \dot\gamma) = 1$. Lastly, we have

$$\frac{d^2}{ds^2}\ell(\gamma_s)|_{s=0} = \frac{d^2}{ds^2}E(\gamma_s)|_{s=0} - \int_0^T \left(g\left(\dot\gamma_s, \frac{\nabla}{ds}\dot\gamma_s\right)|_{s=0}\right)^2 dt.$$

Definition 6.20 (Complete)**.** A manifold M is *complete* if for all points $p \in M$, the exponential map at p is defined on T_pM, the entire tangent space at p.

Definition 6.21 (Open cover)**.** A set $C = \{U_\alpha : \alpha \in A\}$ of subsets U_α of a set X is a cover of X if

$$\bigcup_\alpha \in AU_\alpha \supseteq X.$$

An *open cover* is a cover with each element being an open set. A subcover of a cover of a set is a subset of the cover that also covers the set.

Definition 6.22 (Compact)**.** A topological space is *compact* if every open cover of X has a finite subcover.

7. Bonnet-Myers Theorem

Definition 7.1 (Diameter)**.** The *diameter* of a Riemannian manifold (M, g) is

$$\operatorname{diam}(M, g) = \sup_{p,q\in M} d(p, q).$$

Example 7.1. For example, the 2-sphere,

$$S^1(r) = \{x \in \mathbb{R}^2 : |x| = r\},$$

has

$$\operatorname{diam}(S^1(r), S^1) = \pi r \qquad \text{and} \qquad \operatorname{diam}(S^1(r), \mathbb{R}^2) = 2r.$$

Theorem 7.1 (Bonnet-Myers)**.** Let (M, g) be a complete Riemannian manifold of dimension n whose Ricci curvature satisfies

$$\operatorname{Ric}(g) \geq \frac{n-1}{r^2}$$

for all $v \in SM = \{w \in TM : ||w|| = 1\}$.Then,

$$\operatorname{diam}(M, g) \leq \pi r.$$

Proof. Firstly, note that by Hopf-Rinow, for any $L < \operatorname{diam}(M, g)$ we can find points $p, q \in M$ such that $d(p, q) = L$ and a minimal geodesic $\gamma \in \Omega(p, q)$ with $\ell(\gamma) = d(p, q) = L$. Parameterize $\gamma : [0, L] \to M$ so that $|\dot\gamma| = 1$. Consider some vector field Y along γ such that $Y(p) = 0 = Y(q)$. Since γ is a minimal geodesic, it is a critical point for ℓ. Additionally, since minimal geodesics are critcal points of the length functional , $I(Y, Y)_{[0,L]} \geq 0$. Extend $\dot\gamma(0)$ to an orthonormal basis of T_pM, $\dot\gamma(0) = e_1, e_2, \ldots, e_n$. Let X_i be the vector field such that

$$X_i' = 0 \qquad \text{and} \qquad X_i(0) = e_i.$$

Then, for $i = 1$, $X_1(t) = \dot\gamma(t)$. For $i = 2, \ldots, n$, set

$$Y_i(t) = \sin\left(\frac{\pi t}{L}\right) X_i(t).$$

After using integration by parts

$$I(Y_i, Y_i)_{[0,L]} = -\int_0^L g(Y_i'' + R(\dot\gamma, Y_i)Y_i, \dot\gamma)dt.$$

Since X_i is parallel, this can be written as

$$\int_0^L \sin^2 \frac{\pi t}{L} \left(\frac{\pi^2}{L^2} - R(\dot\gamma, X_i, \dot\gamma, X_i)\right) dt.$$

Since this is length minimizing, it is also non-negative. Note that $R(\dot\gamma, X_1, \dot\gamma, X_1) = 0$. So,

$$\operatorname{Ric}(\dot\gamma, \dot\gamma).$$

From this, it is known that

$$\sum_i^n I(Y_i, Y_i) = \int_0^L \sin^2 \frac{\pi t}{L} \left((n-1)\frac{\pi^2}{L} - \operatorname{Ric}(\dot\gamma, \dot\gamma)\right) dt \geq 0.$$

Recall that by supposition

$$\operatorname{Ric}(\dot\gamma, \dot\gamma) \geq \frac{n-1}{r^2}.$$

From this,

$$\frac{\pi^2}{L^2} \geq \frac{1}{r^2}$$

and

$$L \leq \pi r.$$

By supposition, however, $L < \operatorname{diam}(M, g)$. So,

$$\operatorname{diam}(M, g) \leq \pi r.$$

8. Fundamental Group

Definition 8.1 (Group)**.** A *group* G is a set together with a group operation satisfying the following properties:

- If A and B are two elements in G, then the product AB is also in G.
- The defined multiplication is associative, for all $A, B, C \in G$,
$$(AB)C = A(BC).$$
- There is an identity element I such that $IA = AI = A$ for every element $A \in G$.
- For each element $A \in G$, the set contains an element $B = A^{-1}$ such that $AA^{-1} = A^{-1}A = I$.

Definition 8.2 (Homotopy)**.** A *homotopy* between two functions f and g from a space X to a space Y is a continuous map G from $X \times [0,1] \mapsto Y$ such that $G(x,0) = f(x)$ and $G(1,x) = g(x)$.

Definition 8.3 (Loops)**.** If X is a topological space and $p \in X$, a *loop* is a continuous map
$$L : [0,1] \to X$$
such that $L(0) = p$ and $L(1) = p$.

Definition 8.4 (Fundamental Group)**.** The *fundamental group* of set X is the group formed by the sets of equivalence classes of the set of all loops, i.e., paths with initial and final points at a given basepoint p, under the equivalence relation of homotopy.

Theorem 8.1 (Bonnet-Myers)**.** Let M be a compact Riemannian manifold with positive Ricci curvature; then its fundamental group is finite.

Proof. The goal is to show that the universal covering space $\tilde{M}$ of M is compact. This is due to that fact that if $f : \tilde{M} \to M$ is the covering map, then $f^{-1}(x)$ for $x \in M$ is closed and discrete. By definition, this means it is also finite. So, $\tilde{M}$ is finitely-sheeted over M.

Firstly, any covering space of a smooth manifold can be made into a smooth manifold. Additionally, one can pullback the Riemannian metric g on M via f to get $f * g$ on $\tilde{M}$. Furthermore, due to the fact that f is locally an isometry, it preserves curvature. Moreover, $\tilde{M}$ has Ricci curvature which is bounded below because M does.

Secondly, $\tilde{M}$ is a complete Riemannian manifold. In general, any covering space of a complete Riemannian manifold, with the pulled-back Riemannian metric, is complete. If f is the covering map, a curve γ is a geodesic if and only if $f \circ \gamma$ is one by the local isometry property of f.

Thus, if one starts at a point $\tilde{p} \in \tilde{M}$ and starts a geodesic from $\tilde{p}$, it can be projected to M via f, and extended to a geodesic on $(-\infty, \infty)$ on M by completeness. Additionally, use the covering space property to lift it to $\tilde{M}$ to get a geodesic in $\tilde{M}$. From this, geodesics in $\tilde{M}$ are infinitely extendable, implying completeness. From this, applying the first theorem to $\tilde{M}$ shows compactness and proves the proposed theorem.

References

[Car04] Sean Carroll. Spacetime and geometry: An introduction to general relativity. https://fma.if.usp.br/~mlima/teaching/PGF5292_2021/Carroll_SG.pdf, 2004.

[Gud04] Sigmundur Gudmundsson. An introduction to riemannian geometry. https://www.cis.upenn.edu/~cis6100/Riemann.pdf, 2004.

[Krö10] Klaus Kröncke. Comparison theorems in riemannian geometry. www.google.com/url?sa=i&url=https%3A%2F%2Fphaidra.univie.ac.at%2Fdownload%2Fo%3A1267912&psig=AOvVaw0KZ9kVR47KO4Bp8NVSR42K&ust=1721081690387000&source=images&cd=vfe&opi=89978449&ved=0CAYQrpoMahcKEwiopJSPx6eHAxUAAAAAHQAAAAAQBA, 2010.