Sidorenko's Conjecture

Gilyana Dorzhieva

July 12, 2024

Homomorphism in graphs

Definition

A *homomorphism* from graph H to graph G is a function $f : V(H) \to V(G)$ such that if (u, v) is an edge in H, then $(f(u), f(v))$ is an edge in G. (preserves adjacency)

We can normalize the counting of *homomorphisms* by dividing over the number of all possible mappings from H to G.

Definition

A *homomorphism density* is a probability that a random map of $V(H)$ into $V(G)$ is a homomorphism.

$$t(H, G) = \frac{\hom(H, G)}{|V(G)|^{|V(H)|}}.$$

Important definitions

Definition

A *random graph* $G(n, p)$ (Erdos-Renyi model) is a graph defined by n vertices and probability p that any two vertices will be connected with an edge.

Definition

A *measure space* is a triple $(E, \mathcal{E}, \mu)$, where E is some set. It is the place that we want to 'measure' parts of. (ex. $\mathbb{R}^n$ or a space of all possible outcomes of a dice). $\mathcal{E}$ is called the σ-algebra. A σ-algebra is a set of subsets of E. These will be the sets which we can measure. $\mu : \mathcal{E} \to [0, \infty]$ is the *measure*. The notation is often shortened to only the set and the measure.

What are graphons?

An analogy with rational and real numbers that helps to think about graphons

Example

For $x \in [0,1]$, the minimum of $x^3 - x$ occurs at $x = \frac{1}{\sqrt{3}}$. But if we restrict ourselves in $\mathbb{Q}$, a way to express this minimum is to find a sequence $x_1, x_2, \ldots$ of rational numbers that converges to $\frac{1}{\sqrt{3}}$.

- Consider all graphs as a set of discrete objects (analogously $\mathbb{Q}$), and seek its "completion" (analogously $\mathbb{R}$)
- So the limit object of the sequence of graphs would be a *graphon*
- Graphons can be considered as an analytic generalization of graphs.

Graphon and homomorphism densities

Definition

Graphon is a symmetric measurable function $W\colon [0,1]^2 \to [0,1]$.

Recall homomorphism density. Can we expand it to graphons?

Definition

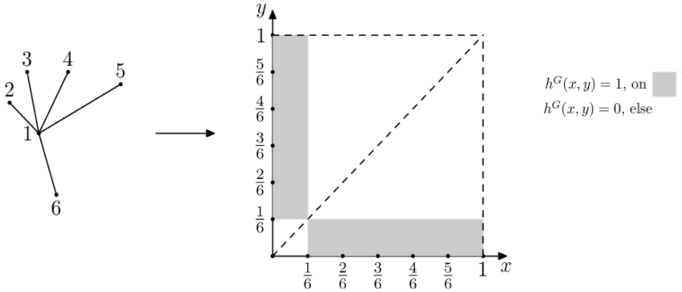
Let G be a graph and W a graphon. The ***G*-density in *W*** is defined to be

$$t(G,W) = \int_{[0,1]^{V(G)}} \prod_{ij \in E(G)} W(x_i, x_j) \prod_{i \in V(G)} dx_i.$$

We also use the same formula when W is a symmetric measurable function.

Graphs to Graphons

We can convert any graph into a graphon, which allow us to start imagining what the limits of some sequences of graph should look like.



Homomorphism into measurable function

The following construction generalizes the homomorphism function. Every bounded function $W : [0,1]^2 \to \mathbb{R}$ defines a graph parameter as follows: For a finite graph F on k nodes, let

$$t(G, W) = \int_{[0,1]^k} \prod_{i,j \in E(G)} W(x_i, x_j)\, dx_1 \ldots dx_k.$$

(We can think of the interval $[0,1]$ as the set of nodes, and of the value $W(x,y)$ as the weight of the edge xy.) While this definition is meaningful for all graphs F, we will mostly use it for simple graphs.

Sidorenko Conjecture

In analytical form:
Denote the Lebesgue measure on $[0,1]$ by μ. Let a function $h(x,y)$ be bounded, non-negative and measurable on $[0,1]^2$. Let G be a bipartite graph where vertices $u_1, u_2, \dots, u_n$ belong to first partition and $v_1, v_2, \dots, v_m$ belong to the second.
Denote by E the set of edges. $|E|$ is the number of edges.

Conjecture

For any bipartite graph G and any function h:

$$\int \prod_{(i,j)\in E} h\left(x_i, y_j\right) d\mu^{n+m} \geq \left(\int h d\mu^2\right)^{|E|}$$

Sidorenko Conjecture

In terms of graph homomorphisms densities:

Conjecture

We say that a graph H satisfies Sidorenko conjecture if for every graph G, and every bipartite graph H,

$$t(H,G) \geq t\left(K_2, G\right)^{e(H)}$$

Question

Can we formulate the conjecture for graphons?

Graphon formulation

Sidorenko's conjecture can be formulated for graphons.

Conjecture

For every bipartite graph G and graphon W,

$$t(G, W) \geq t\left(K_2, W\right)^{e(G)}.$$

How to interpret it?

Conjecture

The inequality states that the random graph with fixed number of vertices and edge density contains the asymptotically minimal number of copies of G over all graphs of the same order and edge density.

$$t(G, W) \geq t\left(K_2, W\right)^{e(G)}$$

Question

When left and right-hand side expressions are equal?

Equality occurs when $W \equiv p$, the constant graphon. The constant graphon corresponds to a random graph of the Erdos-Renyi model, which means the lower bound of the inequality is reached for a random graph.

Proof for Complete Bipartite graphs

Definition

Hölder's inequality says that given $p_1, \dots, p_k \geq 1$ with $1/p_1 + \cdots + 1/p_k = 1$, and real-valued functions $f_1, \dots, f_k$ on a common space, we have

$$\int f_1 f_2 \cdots f_k \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$

where the ***p***-norm of a function f is defined by

$$\|f\|_p := \left(\int |f|^p \right)^{1/p}.$$

Proof for Complete Bipartite graphs

In practice, the case $p_1 = \cdots = p_k = k$ of Hölder's inequality is used often. We can apply Hölder's inequality to show that $K_{s,t}$ is Sidorenko.

Theorem

Complete bipartite graphs are Sidorenko

$$t\left(K_{s,t}, W\right) \geq t\left(K_2, W\right)^{st}$$

Lemma 1

$$t\left(K_{s,1}, W\right) \geq t\left(K_2, W\right)^{s}$$

Lemma 2

$$t\left(K_{s,t}, W\right) \geq t\left(K_{s,1}, W\right)^{t}$$

Proof of Lemma 1.

Proof.

Applying Hölder's inequality: X$=x_1, x_2, \dots x_S$

$$t\left(K_{s,1}, W\right) = \int_{X,y} \prod_{i=1}^{s} W\left(x_i, y\right) = \int_y \left(\int_x W(x,y) \geq \right.$$

$$\geq \left(\int_{x,y} W(x,y)\right)^s = t\left(K_2, W\right)^s$$

∎

Proof of Lemma 2.

Proof.

Similarly to Lemma 2, write the homomorphism density in integral form, apply Holder's inequality and we will get that

$$t\left(K_{s,t}, W\right) \geq t\left(K_{s,1}, W\right)^{t}$$

■

Complying Lemma 1 and Lemma 2 we can see that the inequality

$$t\left(K_{s,t}, W\right) \geq t\left(K_{2}, W\right)^{st}$$

holds true.