SIDORENKO'S CONJECTURE

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Abstract. In this paper we will discuss the Sidorenko's conjecture. It is a conjecture from the field of extremal graph theory that provides an intuitive inequality about the homomorphism densities in graphons. Proofs for a few cases of bipartite graphs such as paths, trees, forests, complete graphs and even cycles will be provided.

1. Introduction

The Sidorenko's conjecture states that if H is a bipartite graph then the random graph with edge density p has in expectation asymptotically the minimum number of copies of H over all graphs of the same order and edge density.

The problem of finding the minimal number of copies of a graph contained in some other graph is related to extremal graph theory.

The conjecture was proposed by Alexander Sidorenko in 1986 and proved for several cases of bipartite graphs such as paths, trees, forests, complete graphs and even cycles. ([Sid92]) Approaches from different mathematical fields were researched in order to work with the conjecture, including entropy and information theory. Later, partial results were obtained for hypercube graphs, more generally, norming graphs, graphs with a vertex complete to all others. However, the conjecture still remains open for the general case of all bipartite graphs.

In the Introduction section, we will formulate the conjecture, then in Preliminaries we will explore the concept of homomorphism counting and its analytical representation, graphons and its connection to the conjecture. In section 3 we will discuss the analytical inequalities for bipartite graphs and further will prove them for specific types of graphs in section 4.

Conjecture 1.1. *The conjecture states that for every graph G, and every bipartite graph H,*

$$t(H, G) \geq t\left(K_2, G\right)^{e(H)}$$

Alternatively, we can extend the conjecture to graphons. For every bipartite graph H and graphon W,

$$t(H, W) \geq t\left(K_2, W\right)^{e(H)}.$$

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Finally, expressing homomorphism densities in terms of integral inequality, the conjecture has the following form. For any bipartite graph G and any function h:

$$\int \prod_{(i,j)\in E} h\left(x_i, y_j\right) d\mu^{n+m} \geq \left(\int h d\mu^2\right)^{|E|}$$

Denote the Lebesgue measure on $[0,1]$ by μ. Let a function $h(x,y)$ be bounded, non-negative and measurable on $[0,1]^2$.

The conjecture states that if H is a bipartite graph then the random graph with edge density p has in expectation asymptotically the minimum number of copies of H over all graphs of the same order and edge density.

Remark 1. The number of homomorphisms from H to G does not exactly correspond to the number of copies of subgraphs H inside G, because the homomorphisms can be non-injective. However, since the number of non-injective homomorphisms contribute at most $O_H\left(n^{|V(H)-1|}\right)$ (where $n = |V(G)|$), they make a lower order contribution compared to all homomorphisms($n^{V(H)}$) as $n \to \infty$ when H is fixed. [Zha23]

Indeed, as we set the graphon W to be a constant graphon, $W \equiv p$, for some constant $p \in [0,1]$, the equality between left and right-hand sides occurs.

2. Preliminaries

Definition 2.1. A *homomorphism* from H to G is a function $f : V(H) \to V(G)$ such that if (u,v) is an edge in H, then $(f(u), f(v))$ is an edge in G, in other words, a function that preserves adjacency.

Definition 2.2. Define the homomorphism number $hom(F,G)$ to be the number of homomorphisms from F to G.

We can normalize the counting of *homomorphisms* by dividing over the number of all possible mappings from H to G.

Definition 2.3. A *homomorphism density* is a probability that a random map of $V(H)$ into $V(G)$ is a homomorphism.

$$t(H,G) = \frac{\hom(H,G)}{|V(G)|^{|V(H)|}}.$$

Definition 2.4. A *random graph* $G(n,p)$ (Erdős-Rényi model) is a graph defined by n vertices and probability p that any two vertices will be connected with an edge.

Graphons, short for graph functions, are the limiting objects for sequences of large, finite graphs, which reflect properties of the finite large graphs, and vice versa.

Definition 2.5. *Graphon* is a symmetric measurable function W: $[0,1]^2 \to [0,1]$.

Graphons can be considered as an analytic generalization of graphs.

Definition 2.6. A *constant graphon*, $W(x,y) \equiv p$, for some constant $p \in [0,1]$ corresponds to the random graph $G(n,p)$, the Erdős-Rényi model.

Homomorphism densities defined earlier can be naturally expanded to graphons.

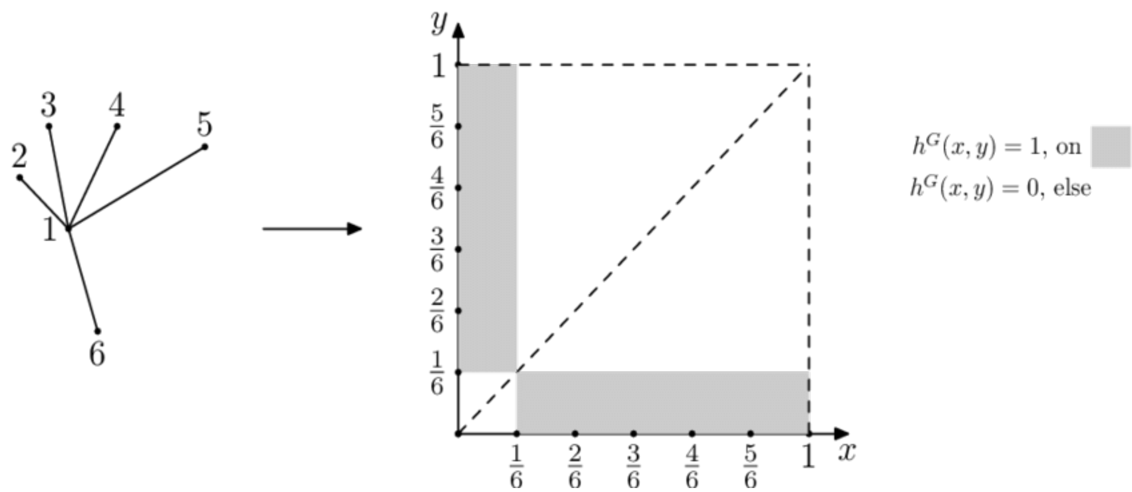
Definition 2.7. Let F be a graph, and let W be a graphon. Define the homomorphism density

$$t(F, W)=\int_{[0,1]^{V(F)}} \prod_{ij \in \mathrm{E}(F)} W\left(x_i, x_j\right) \prod_{i \in \mathrm{V}(F)} d x_i .$$

Note that in this context, the injective homomorphism density is insignificant because when randomly assigning vertices i and j to x_i and x_j in the interval $[0,1], x_i \neq x_j$ with probability 1 . Thus, the injective homomorphism density and homomorphism density are essentially equivalent in this context.

In the context of the conjecture, it is important to mention that any graph can be converted into a graphon, thus, we can use the homomorphism density Definition 2.7 for graphs.

Definition 2.8. Given a graph G with n vertices (labeled $1, \ldots, n$), we define its associated graphon as $W_G:[0,1]^2 \rightarrow[0,1]$ obtained by partitioning $[0,1]=I_1 \cup I_2 \cup \cdots I_n$ with $\lambda\left(I_i\right)=1 / n$ such that if $(x, y) \in I_i \times I_j$, then $W(x, y)=1$ if i and j are connected in G and 0 otherwise. (Here $\lambda(I)$ is the Lebesgue measure of I.)
See example below from [HMRS17].



Definition 2.9. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathrm{P})$, where Ω is a sample space, $\mathcal{F}$ is a sigma-algebra of events and P is a probability measure on $\mathcal{F}$.

- the sample space Ω is the set of all possible outcomes of a probabilistic experiment
- the sigma-algebra $\mathcal{F}$ is the collection of all subsets of Ω to which we will assign probabilities; these subsets are called events
- the probability measure P is a function that associates a probability to each of the events belonging to the sigma-algebra $\mathcal{F}$.

Definition 2.10. A measure space is a triple $(X, \mathcal{A}, \mu)$ where X is a set, $\mathcal{A}$ a σ-algebra of its subsets, and $\mu: \mathcal{A} \rightarrow[0,+\infty]$ a measure. A measure space consists of a measurable space and a measure. The notation $(X, \mathcal{A}, \mu)$ is often shortened to (X, μ) and one says that μ is a measure on X.

Definition 2.11. Hölder's inequality says that given $p_1, \ldots, p_k \geq 1$ with $1/p_1 + \cdots + 1/p_k = 1$, and real-valued functions $f_1, \ldots, f_k$ on a common space, we have

$$\int f_1 f_2 \cdots f_k \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$

where the $\boldsymbol{p}$-norm of a function f is defined by

$$\|f\|_p := \left(\int |f|^p \right)^{1/p}.$$

Theorem 2.12. *Fubini's Theorem*

Suppose A and B are complete measure spaces. Suppose $f(x, y)$ is $A \times B$ measurable. If

$$\int_{A \times B} |f(x, y)| \mathrm{d}(x, y) < \infty$$

where the integral is taken with respect to a product measure on the space over $A \times B$, then

$$\int_A \left(\int_B f(x, y) \mathrm{d}y \right) \mathrm{d}x = \int_B \left(\int_A f(x, y) \mathrm{d}x \right) \mathrm{d}y = \int_{A \times B} f(x, y) \mathrm{d}(x, y)$$

the first two integrals being iterated integrals with respect to two measures, respectively, and the third being an integral with respect to a product of these two measures.

The Hölder's inequality and Fubini's theorem will play an important role when working with integral inequalities.

3. Analytical inequalities for bipartite graphs

In this section we will discuss the analytical inequalities for bipartite graphs obtained by A. Sidorenko in [Sid92]. The following integral inequalities represent homomorphism densities in bipartite graphs.

Definition 3.1. Let $\Omega = (X, \mu)$ be a measure space where μ is a finite σ-additive measure defined on a σ-algebra of subsets of the set X. We denote by $K(\Omega)$ the class of non-negative, bounded and measurable functions on X. Let $K_+(\Omega)$ be the subclass of functions from $K(\Omega)$ which are strictly positive almost everywhere (with respect to the measure). $K(\Omega)$ is closed under products.

For a pair of such spaces $\Omega = (X, \mu), \Lambda = (Y, \nu)$, we will use functions from $K(\Omega \otimes \Lambda)$ to construct functions which belong to the class $K(\Omega^m \otimes \Lambda^n)$. Since the considered functions are bounded and measurable, Fubini's theorem can be applied. Fubini's theorem allows to change the order of integration for multiple integrals.

Example 1. The product of four functions which are copies of the same function $h \in K(\Omega \otimes \Lambda)$ but have different pairs of the arguments:

$$h(x_1, y_1) h(x_2, y_1) h(x_3, y_1) h(x_3, y_2).$$

This product is a function with arguments $x_1, x_2, x_3 \in X, y_1, y_2 \in Y$ and belongs to the class $K(\Omega^3 \otimes \Lambda^2)$. We can think of function h as an indicator of the presence of an edge between two vertices x_i, y_i. The vertices correspond to the arguments x_1, x_2, x_3, y_1, y_2, and the edges correspond to the factors of the product. This graph is bipartite, where $x_i \in X$ and $y_i \in Y$, and no two vertices from the same set are adjacent.

Definition 3.2. In general, a bipartite graph is a triple $G = (V_1, V_2, E)$, where V_1, V_2 are disjoint finite sets, and $E \subseteq V_1 \times V_2$. The elements of V_i are called vertices of colour $i (i = 1, 2)$, and the elements of E are called edges.

For instance,

$$V_1 = \{u_1, u_2, u_3\}, \quad V_2 = \{w_1, w_2\}, \quad E = \{(u_1, w_1), (u_2, w_1), (u_3, w_1), (u_3, w_2)\}$$

define a graph which is isomorphic to the graph in Fig. 1.

The product of function $h(x, y)$ can be visualized as a diagram. We say that any bipartite graph can be regarded as a diagram. For any function $h \in K(\Omega \otimes \Lambda)$, a graph $G = (\{u_1, \dots, u_m\}, \{w_1, \dots, w_n\}, E)$ generates the function

$$\prod_{(u_i, w_j) \in E} h(x_i, y_j)$$

which belongs to the class $K(\Omega^m \otimes \Lambda^n)$ and has the arguments $x_1, \dots, x_m, y_1, \dots, y_n$. Sidorenko obtains a number of integral inequalities for such functions. One of them is

$$\int \prod_{(u_i, w_j) \in E} h(x_i, y_j) \, \mathrm{d}\mu^m \, \mathrm{d}\nu^n \geq \left(\int h(x, y) \mathrm{d}\mu \mathrm{d}\nu \right)^{|E|} \mathrm{d}\mu(X)^{m-|E|} \mathrm{d}\nu(Y)^{n-|E|}. \tag{3.1}$$

The left-hand side of inequality can be considered as a functional on $K(\Omega \otimes \Lambda)$ defined by the graph G.

The Sidorenko conjecture states that this inequality holds for any bipartite graph G, any spaces Ω, Λ, and any function $h \in K(\Omega \otimes \Lambda)$. The conjecture can not yet be proven completely but it was proved for few types of graphs G.

Now we will discuss inequality (1) along with more general inequalities.

Denote by $\mathcal{F}$ the class of bipartite graphs $G = (\{u_1, \dots, u_m\}, \{w_1, \dots, w_n\}, E)$ which satisfy the following conditions:

Condition A. $|E| \geq m, |E| \geq n$.

Condition B. For any spaces Ω, Λ and any functions $h \in K(\Omega \otimes \Lambda)$, $f, f_1, \dots, f_m \in K(\Omega), g, g_1, \dots, g_n \in K(\Lambda)$

(3.2)

$$\int \prod_{(u_i, w_j) \in E} h(x_i, y_j) \prod_{i=1}^{m} f_i(x_i) \prod_{j=1}^{n} g_j(y_j) \, \mathrm{d}\mu^m \, \mathrm{d}\nu^n \left(\int f(x) \mathrm{d}\mu \right)^{|E|-m} \left(\int g(y) \mathrm{d}\nu \right)^{|E|-n}$$

$$\geq \left(\int h(x, y) \left(f(x)^{|E|-m} g(y)^{|E|-n} \prod_{i=1}^{m} f_i(x) \prod_{j=1}^{n} g_j(y) \right)^{1/|E|} \mathrm{d}\mu \mathrm{d}\nu \right)^{|E|}.$$

We should notice that (3.1) is a particular case of (3.2) when functions $f, f_1, \ldots, f_m, g, g_1, \ldots g_n$ are constants equal to the unit. However, the domain of definition is wider for inequality (3.1), since Condition A is not required (see Remark 1 below). There are other cases of inequality (3.2). If we set

$$f(x)=\left(\int h(x,y)\left(g(y)^{|E|-n}\prod_{j=1}^{n} g_j(y)\right)^{1/|E|} \mathrm{d}\nu\right)^{|E|/m}\left(\prod_{i=1}^{m} f_i(x)\right)^{1/m},$$

the integral in the second factor of the left-hand side of (3.2) coincides with the integral in the right-hand side. It gives the inequality

$$\begin{aligned}(3.3)\quad &\int \prod_{(u_i,w_j)\in E} h\left(x_i,y_j\right)\prod_{i=1}^{m} f_i\left(x_i\right)\prod_{j=1}^{n} g_j\left(y_j\right)\mathrm{d}\mu^m\ \mathrm{d}\nu^n\left(\int g(y)\mathrm{d}\nu\right)^{|E|-n}\\ &\geq\left(\int\left(\int h(x,y)\left(g(y)^{|E|-n}\prod_{j=1}^{n} g_j(y)\right)^{1/|E|} \mathrm{d}\nu\right)^{|E|/m}\left(\prod_{i=1}^{m} f_i(x)\right)^{1/m} \mathrm{d}\mu\right)^{m}.\end{aligned}$$

In its turn, (3.3) and Hölder's inequality (using $|E|\geq m$) imply (3.2). Therefore, (3.2) and (3.3) are equivalent. By analogy, another equivalent form of (3.2) is

$$\begin{aligned}(3.4)\quad &\int \prod_{(x_i,w_j)\in E} h\left(x_i,y_j\right)\prod_{i=1}^{m} f_i\left(x_i\right)\prod_{j=1}^{n} g_j\left(y_j\right)\mathrm{d}\mu^m\ \mathrm{d}\nu^n\left(\int f(x)\mathrm{d}\mu\right)^{|E|-m}\\ &\geq\left(\int\left(\int h(x,y)\left(f(x)^{|E|-m}\prod_{i=1}^{m} f_i(x)\right)^{1/|E|} \mathrm{d}\mu\right)^{|E|/n}\left(\prod_{j=1}^{n} g_j(y)\right)^{1/n} \mathrm{d}\nu\right)^{n}.\end{aligned}$$

In particular, setting all functions $f, f_1, \ldots, f_m, g, g_1, \ldots, g_n$ equal to the constant 1 in (3.3) and (3.4), we get the inequalities

$$(3.5)\quad \int \prod_{(w_i,w_j)\in E} h\left(x_i,y_j\right)\mathrm{d}\mu^m\ \mathrm{d}\nu^n \geq\left(\int\left(\int h(x,y)\mathrm{d}\nu\right)^{|E|/m} \mathrm{d}\mu\right)^{m}\nu(Y)^{n-|E|},$$

$$(3.6)\quad \int \prod_{(w_i,w_j)\in E} h\left(x_i,y_j\right)\mathrm{d}\mu^m\ \mathrm{d}\nu^n \geq\left(\int\left(\int h(x,y)\mathrm{d}\mu\right)^{|E|/n} \mathrm{d}\nu\right)^{n}\mu(X)^{m-|E|},$$

which are stronger than (3.1).

Remark 2. Let graph G belong to the class $\mathcal{F}$. Then inequality (3.3) holds. For proving the future theorems in this paper, we would want for inequality (3.3) be true for a graph G when it is extended by adding any number of isolated vertices to it. Applying Hölder's inequality we can check that the inequality (3.3) remains valid for this condition. Since inequality (3.2) is equivalent to (3.3) with $m \le |E|$ and to inequality (3.4) with $n \le |E|$, the property 'belongs to the class $\mathcal{F}$ is preserved when any number of isolated vertices of both colours is added (provided that Condition A is not violated). Finally, inequality (3.1) for a graph $G \in \mathcal{F}$ immediately implies that the same inequality is valid for extensions of G by isolated vertices.

4. Sidorenko's conjecture for special cases of bipartite graphs

Dealing with bipartite graphs, we will use the following definitions and notation.

4.1. **Definitions and notations.**

For a bipartite graph $G = (V_1, V_2, E)$, denote $v_i(G) = |V_i|\ (i = 1, 2), e(G) = |E|$.

Definition 4.1. If $E = V_1 \times V_2$, the graph G is called complete bipartite and is denoted by $K_{m,n}$ where $m = v_1(G), n = v_2(G)$.

Definition 4.2. A simple graph is an undirected graph without weights, multiple edges, or loops.

Definition 4.3. Graphs are called independent if they have no common vertices.

Definition 4.4. Let $G' = (V_1', V_2', E')$ and $G'' = (V_1'', V_2'', E'')$ be independent, then graphs $G' + G''$ and $G' \times G''$ are defined as follows:

$$G' + G'' = (V_1' \cup V_1'', V_2' \cup V_2'', E' \cup E''),$$
$$G' \times G'' = (V_1' \cup V_1'', V_2' \cup V_2'', E' \cup E'' \cup (V_1' \times V_2'') \cup (V_1'' \times V_2')).$$

Let $k = k(G)$ be the maximal integer such that the bipartite graph G can be represented as $G = G_1 + G_2 + \ldots + G_k$. It is easy to see that this maximal representation is unique. The independent graphs $G_1, G_2, \ldots, G_k$ in the representation are called connected components of G. If $k(G) = 1$, the graph G is called connected.

Definition 4.5. A *tree* is a connected bipartite graph whose total number of vertices exceeds the number of edges by 1 (i.e., $v_1(G) + v_2(G) = e(G) + 1$).

Definition 4.6. A graph is called a *forest* if all of its connected components are trees.

4.2. **Theorems.**

According to the definition, a bipartite graph belongs to the class $\mathcal{F}$ if it satisfies Conditions A, B. The following theorems obtained and proved by A. Sidorenko in [Sid92] will be formulated, and their proofs presented in next subsection.

Theorem 4.7. *Let a graph G satisfy Condition A. If $v_1(G) \le 3$ or $v_2(G) \le 3$ then $G \in \mathcal{F}$.*

Theorem 4.8. *Let a bipartite graph G'' be obtained from a graph G by adding a new vertex and a new edge which joins this vertex to a vertex a of the graph G. If G belongs to $\mathcal{F}$ then G'' also belongs to $\mathcal{F}$.*

Corollary 1. *Any tree with more than one vertex belongs to the class $\mathcal{F}$. The definition of the class $\mathcal{F}$ is symmetric with respect to the colours of vertices. This implies the following assertions.*

Theorem 4.9. *If independent bipartite graphs G' and G'' belong to $\mathcal{F}$, then $G' + G''$ also belongs to $\mathcal{F}$.*

Taking into account Remark 2, we get the following corollary.

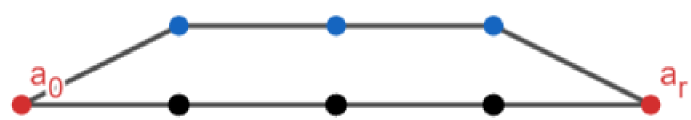
Corollary 2. *If a graph satisfies Condition A and all of its connected components belong to $\mathcal{F}$, then the graph belongs to $\mathcal{F}$ as well.*

It can be deduced from theorems 4.8 and 4.9 that all *trees* (which include *paths*) and *forests* (that satisfy Condition A) belong to the class $\mathcal{F}$.

Theorem 4.10. *If $H \in \mathcal{F}$ then $H \times K_{p,q} \in \mathcal{F}$ with any $p, q \in \{0, 1, 2, \ldots\}$.*

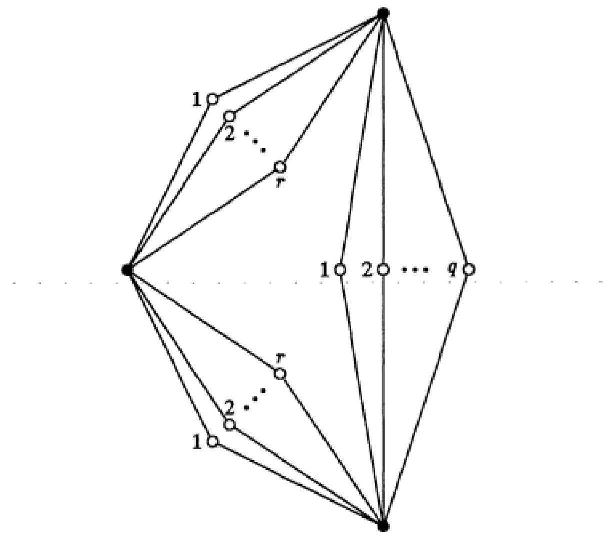
Theorem 4.10 implies that all complete bipartite graphs belong to the class $\mathcal{F}$.

Theorem 4.11. *Let a graph G belong to $\mathcal{F}$. Let us mark some of its vertices (their colours are not important) such that each edge has at most one marked end. Now we take k independent copies of G and, for each marked vertex, we identify (glue) all of its k images. Then the resulting graph G' belongs to the class $\mathcal{F}$.*



Example 2. See image above. A path P_τ of length r is a bipartite graph with vertices $a_0, \ldots, a_r$ where a_j has colour 1 if j is even, or colour 2 if j is odd; the edges are $(a_0, a_1), (a_2, a_1), (a_2, a_3), \ldots,$ and the last edge in this sequence is (a_{r-1}, a_r) if r is odd, or (a_r, a_{r-1}) if r is even. Clearly, this graph is a tree, and according to Corollary 1, belongs to the class $\mathcal{F}$. Let us apply Theorem 4.11 with $G = P_r, k = 2$, choosing a_0 and a_r as marked vertices. The resulting graph G' is connected and has $2r$ vertices; each of them is an end of two edges. Such a graph is called a cycle of length $2r$, and according to Theorem 4.11 , it belongs to the class $\mathcal{F}$.

Theorem 4.12. *Let us consider the graphs G and G' from the formulation of Theorem 4.11. Let us choose an unmarked vertex a of colour i in the graph G. Let us consider a graph G_1 which belongs to $\mathcal{F}$ and satisfies the inequality $v_i(G_1) \leq k$. We identify (glue) each of its vertices of colour i with one of the k images of the vertex a in the graph G'. Then the resulting graph G'' belongs to the class $\mathcal{F}$.*



Example 3. See image above. As mentioned before, Theorem 4.10 yields that all complete bipartite graphs belong to the class $\mathcal{F}$. Let $r \geq 1$. Choose one of the vertices of colour 1 in the complete bipartite graph $K_{2,r}$ as a marked vertex and the other as the vertex a. Now apply Theorem 4.12 with $G = K_{2,r}, k = 2, G_1 = K_{2,q}$. The resulting graph G'' is denoted by $T_{r,r,q}$ and is shown in image above. It has three vertices of colour 1 and $2r + q$ vertices of colour 2.

Finally, we formulate necessary and sufficient conditions for the equality in (1) when G is a tree or a forest.

Theorem 4.13. *Let a bipartite graph G be a forest. The equality in (1) is attained if and only if the two following conditions hold simultaneously:*

(a) *if $v_2(G) > 1$ then the function $\varphi(x) = \int h(x, y)\mathrm{d}\nu(y)$ is equal to a constant for almost all x with respect to the measure μ;*

(b) *if $v_1(G) > 1$ then the function $\psi(y) = \int h(x, y)\mathrm{d}\mu(x)$ is equal to a constant for almost all y with respect to the measure ν.*

4.3. **Proofs.**

We will start with the proof of Theorem 4.12, and Theorems 4.8, 4.10 and 4.11 will be obtained as special case of Theorem 4.12. Finally we will prove Theorems 4.7 and 4.13.

Proof. Theorem 4.12

Without loss of generality, we may assume that the marked vertices of colour 1 are $u_1, u_2, \ldots, u_{s_1}$, and the marked vertices of colour 2 are $w_1, w_2, \ldots, w_{s_2}$. For definiteness, let the vertex a of the graph G be a vertex u_0 of colour 1 , and its copies in G' are

$u_0^1, u_0^2, \ldots, u_0^k$. Since, in the general case, G_1 may have less than k vertices of colour 1 , let us denote by G_2 the graph obtained from G_1 by adding $k - v_1(G_1)$ isolated vertices of colour 1. According to Remark 1 from Section 2, inequality (3) is valid for G_2.

Notice that every edge of G produces exactly k edges in the graph G', since there is no edge with two marked ends. Let us pick up functions $f \in K_+(\Omega), g \in K_+(\Lambda)$, and functions which correspond to the vertices of the graph G''. Then the configuration product can be viewed as the product of the following $k+2$ factors: A_i, the product of the functions which correspond to the edges and the unmarked vertices of the i th copy of $G(i = 1, 2, \ldots k)$; A_0, the product of the functions which correspond to the vertices of colour 2 and to the edges of G_2; B_0, the product of the functions which correspond to the marked vertices. Therefore, $B_0 A_1 A_2 \ldots A_k A_0$ is the configuration product for G''. Denote by $D_1(y)$ the product of the functions $g_j(y)$ which correspond to the vertices of colour 2 of G', and denote by $D_2(y)$ the product of the functions which correspond to the vertices of colour 2 of G_2. Let $D(y)$ be the product of all the functions which correspond to the vertices of colour 2 of G'' that is $D(y) = D_1(y) D_2(y)$.

Denote by $B_i(x_0^i, \ldots)$ the integral of A_i over all the variables which correspond to the unmarked vertices of the i th copy of G, except the vertex u_0^i that corresponds to the variable x_0^{i}.

Now we fix the values of the variables which correspond to the marked vertices and consider B_i as a function of x_0^i. Then $B_1 B_2 \ldots B_k A_0$ is the configuration product for the graph G_2 where B_i is a function which corresponds to the vertex u_0^i. Applying inequality (3) for G_2, we obtain:

$$\int B_0 A_1 \dots A_k A_0 \, \mathrm{d}\mu^{v_1(G'')} \mathrm{d}\nu^{v_2(G'')} \left(\int g \, \mathrm{d}\nu \right)^{e(G_2)-v_2(G_2)}$$

$$= \int B_0 A_0 \left(\int A_1 \dots A_k \, \mathrm{d}\mu^{k(v_1(G')-s_1-1)} \mathrm{d}\nu^{k(v_2(G')-s_2)} \right) \mathrm{d}\mu^{s_1+k} \, \mathrm{d}\nu^{s_2+v_2(G_2)}$$

$$\times \left(\int g \, \mathrm{d}\nu \right)^{e(G_2)-v_2(G_2)}$$

$$= \int B_0 A_0 B_1 \dots B_k \, \mathrm{d}\mu^{s_1+k} \, \mathrm{d}\nu^{s_2+v_2(G_2)} \left(\int g \, \mathrm{d}\nu \right)^{e(G_2)-v_2(G_2)}$$

$$= \int B_0 \left(\int A_0 B_1 \dots B_k \, \mathrm{d}\mu^{k} \, \mathrm{d}\nu^{v_2(G_2)} \right) \left(\int g \, \mathrm{d}\nu \right)^{e(G_2)-v_2(G_2)} \mathrm{d}\mu^{s_1} \, \mathrm{d}\nu^{s_2} \tag{4.1}$$

$$\geq \int B_0 \left(\int \left(\int h(x,y) \left(g(y)^{e(G_2)-v_2(G_2)} D_2(y) \right)^{1/e(G_2)} \mathrm{d}\nu \right)^{e(G_2)/k} \right.$$

$$\left. \times \left(\prod_{i=1}^{k} B_i(x, \dots) \right)^{1/k} \mathrm{d}\mu(x) \right)^{k} \mathrm{d}\mu^{s_1} \, \mathrm{d}\nu^{s_2}$$

$$= \int B_0 \left(\int f_0(x) \left(\prod_{i=1}^{k} B_i(x, \dots) \right)^{1/k} \mathrm{d}\mu(x) \right)^{k} \mathrm{d}\mu^{s_1} \, \mathrm{d}\nu^{s_2},$$

where

$$f_0(x) = \left(\int h(x,y) \left(g(y)^{e(G_2)-v_2(G_2)} D_2(y) \right)^{1/e(G_2)} \mathrm{d}\nu \right)^{e(G_2)/k}.$$

Denote by A the geometric mean of the functions $A_1, \dots, A_k$ where, for each unmarked vertex b of G, instead of k different variables which correspond to k copies of b in G', we plug the only variable which corresponds to b. By Hölder's inequality,

$$\left(\prod_{i=1}^{k} B_i(x, \dots) \right)^{1/k} \geq \int A \, \mathrm{d}\mu^{v_1(G)-s_1-1} \, \mathrm{d}\nu^{v_2(G)-s_2}.$$

Thus

$$\int f_0(x) \left(\prod_{i=1}^{k} B_i(x, \dots) \right)^{1/k} \mathrm{d}\mu(x) \geq \int f_0(x) A \, \mathrm{d}\mu^{v_1(G)-s_1} \, \mathrm{d}\nu^{v_2(G)-s_2}. \tag{4.2}$$

Applying Hölder's inequality again, we obtain

(4.3)

$$\int B_0\left(\int f_0\left(x_0\right)A\ \mathrm{d}\mu^{v_1(G)-s_1}\ \mathrm{d}\nu^{v_2(G)-s_2}\right)^k\ \mathrm{d}\mu^{s_1}\ \mathrm{d}\nu^{s_2}\left(\int f\ \mathrm{d}\mu\right)^{(k-1)s_1}\left(\int g\ \mathrm{d}\nu\right)^{(k-1)s_2}$$

$$=\int B_0\left(\int f_0\left(x_0\right)A\ \mathrm{d}\mu^{v_1(G)-s_1}\ \mathrm{d}\nu^{v_2(G)-s_2}\right)^k\ \mathrm{d}\mu^{s_1}\ \mathrm{d}\nu^{s_2}\left(\int\prod_{i=1}^{s_1}f\left(x_i\right)\prod_{j=1}^{s_2}g\left(y_j\right)\mathrm{d}\mu^{s_1}\ \mathrm{d}\nu^{s_2}\right)^{k-1}$$

$$\geq\left(\int\left(B_0\right)^{1/k}f_0\left(x_0\right)A\left(\prod_{i=1}^{s_1}f\left(x_i\right)\prod_{j=1}^{s_2}g\left(y_j\right)\right)^{(k-1)/k}\mathrm{d}\mu^{v_1(G)}\mathrm{d}\nu^{v_2(G)}\right)^k$$

We may regard

$$\left(B_0\right)^{1/k}f_0\left(x_0\right)\left(\prod_{i=1}^{s_1}f\left(x_i\right)\prod_{j=1}^{s_2}g\left(y_j\right)\right)^{(k-1)/k}$$

as the product of functions which correspond to the vertices $u_0, u_1, \ldots, u_{s_1}, w_1, \ldots, w_{s_2}$ of G. Then, multiplying by the function A, we get the configuration product for G. Denote by $C(x)$ the product of all the functions $f_i(x)$ which correspond to the vertices of colour 1 in the graph G''. Applying inequality (2) to the integral of the configuration product of G, we have

$$\int A\left(B_0\right)^{1/k}f_0\left(x_0\right)\left(\prod_{i=1}^{s_1}f\left(x_i\right)\prod_{j=1}^{s_2}g\left(y_j\right)\right)\mathrm{d}\mu^{v_1(G)}\mathrm{d}\nu^{v_2(G)}\left(\int f\ \mathrm{d}\mu\right)^{e(G)-v_1(G)}\left(\int g\ \mathrm{d}\nu\right)^{e(G)-v_2(G)}$$

$$\geq\left(\int h(x,y)\left(f(x)^\alpha g(y)^\beta f_0(x)C(x)^{1/k}D_1(y)^{1/k}\right)^{1/e(G)}\mathrm{d}\mu\mathrm{d}\nu\right)^{e(G)}$$

$$=\left(\int\left(f(x)^\alpha C(x)^{1/k}f_0(x)\left(\int h(x,y)\left(g(y)^\beta D_1(y)^{1/k}\right)^{1/e(G)}\mathrm{d}\nu\right)^{e(G)}\right)^{1/e(G)}\mathrm{d}\mu\right)^{e(G)},$$

where

$$\alpha=e(G)-v_1(G)+\frac{k-1}{k}s_1,\quad\beta=e(G)-v_2(G)+\frac{k-1}{k}s_2.$$

Substituting the expression for f_0 and applying Hölder's inequality, we get

$$
\begin{aligned}
&f_0(x)\left(\int h(x,y)\left(g(y)^{\beta}D_1(y)^{1/k}\right)^{1/e(G)}\mathrm{d}\nu\right)^{e(G)}\\
&=\left(\int h(x,y)\left(g(y)^{e(G_2)-v_2(G_2)}D_2(y)\right)^{1/e(G_2)}\mathrm{d}\nu\right)^{e(G_2)/k}\\
&\times\left(\int h(x,y)\left(g(y)^{\beta}D_1(y)^{1/k}\right)^{1/e(G)}\mathrm{d}\nu\right)^{e(G)}\\
&\geq\left(\int h(x,y)g(y)^{\beta'}D(y)^{1/e(G'')}\mathrm{d}\nu\right)^{e(G'')/k},
\end{aligned}
\tag{4.4}
$$

where

$$
\beta'=\frac{\beta+\left(e\left(G_2\right)-v_2\left(G_2\right)\right)/k}{e\left(G''\right)/k}=\frac{e\left(G''\right)-v_2\left(G''\right)}{e\left(G''\right)}.
$$

Now we again apply Hölder's inequality:

(4.5)

$$
\left(\int\left(f(x)^{\alpha}C(x)^{1/k}\left(\int h(x,y)g(y)^{\beta'}D(y)^{1/e(G'')}\mathrm{d}\nu\right)^{e(G'')/k}\right)^{1/e(G)}\mathrm{d}\mu\right)^{ke(G)}\left(\int f\ \mathrm{d}\mu\right)^{e(G_2)}
$$

$$
\geq\left(\int f(x)^{\alpha'}C(x)^{1/e(G'')}\left(\int h(x,y)g(y)^{\beta'}D(y)^{1/e(G'')}\mathrm{d}\nu\right)\mathrm{d}\mu\right)^{e(G'')}
$$

where

$$
\alpha'=\frac{\alpha k+e\left(G_2\right)}{e\left(G''\right)}=\frac{e\left(G''\right)-v_1\left(G''\right)}{e\left(G''\right)}.
$$

Combining inequalities (4.1)-(4.5), we get inequality (2) for the graph G''. □

Proof. Theorem 4.8

Apply Theorem 4.12 with k=1, $G_1=K_{1,1}$ □

Proof. Theorem 4.9

Let $G=G'+G''$. The integral of the configuration product of G is equal to the product of the integrals of the configuration products of G' and G''. The product of the left-hand sides of inequality (3.2) for G' and G'' is equal to the left-hand side of the same inequality for G. On the other hand, according to Hölder's inequality, the product of the right-hand sides of inequality (3.2) for G' and G'' is greater than or equal to the right-hand side of the same inequality for G. Therefore, inequality (3.2) for G is proved. □

Proof. Theorem 4.10 According to Corollary 1, $K_{1,q},K_{p,1}\in\mathcal{F}$. Thus Theorem 4.12 with $G=K_{1,q},k=v_1(H),i=1,G_1=H$ yields $G''\in\mathcal{F}$. Notice that $G''=H\times K_{0,q}$. Now apply Theorem 4.12 with $G=K_{p,1},k=v_2\left(H\times K_{0,q}\right),i=2,G_1=H\times K_{0,q}$ to get $H\times K_{p,q}=\left(H\times K_{0,q}\right)\times K_{p,0}\in\mathcal{F}$. □

Proof. Theorem 4.7

We use induction on the total number of vertices of G. The base of the induction with $v_1(G) = v_2(G) = e(G) = 1$ is trivial. Let us prove the induction step: assume that any graph with less than l vertices belongs to the class $\mathcal{F}$, and consider a bipartite graph G with l vertices where $l \geq 3, l = m + n, v_1(G) = m, v_2(G) = n,\ m \leq 3$. If G is not connected, we may apply the hypothesis of induction and Corollary 2. If G has a vertex which is an end of one edge only, we may apply the hypothesis of induction and Theorem 4.8. Thus we assume that G is connected and any vertex belongs to at least two edges. Denote by s_i the number of vertices of colour 2 which are adjacent to exactly i vertices of colour $1, i = 0, 1, \ldots, m$. In particular, $s_0 = s_1 = 0$. If $s_m > 0$, then G can be represented as $G = G' \times K_{0,1}$ and we may apply the hypothesis of induction and Theorem 4.10. So we assume $s_m = 0$. Hence, $e(G) = 2s_2 + \ldots + (m-1)s_{m-1}$. Since $e(G) > 0$ and $m \leq 3$, we have $m = 3$ and $s_2 = n$. Let u_1, u_2, u_3 be vertices of colour 1. The vertices of colour 2 are divided into subsets W_{12}, W_{13}, W_{23}, where $W_{\alpha\beta}$ consists of the vertices adjacent to u_α and u_β. Among $|W_{12}|, |W_{13}|, |W_{23}|$, one may find either a pair of even or a pair of odd numbers. Let, for definiteness, $|W_{12}| + |W_{13}| = 2r$, where r is an integer. As the vertex u_1 is not isolated, $r > 0$. Set $q = |W_{23}|$. Without loss of generality, one may assume

$$W_{12} = \{w_1, \ldots, w_s\}, \quad W_{13} = \{w_{s+1}, \ldots, w_{2r}\}, \quad W_{23} = \{w_{2r+1}, \ldots, w_{2r+q}\}.$$

Let us pick up functions $h \in K(\Omega, \Lambda), f, f_1, f_2, f_3 \in K_+(\Omega), g, g_1, \ldots, g_{2r+q} \in K_+(\Lambda)$ and denote

$$F_{12}(x_1, x_2) = \int \prod_{j=1}^{s} \left(h(x_1, y_j)\, h(x_2, y_j)\, g_j(y_j)\right) \mathrm{d}\nu^s,$$

$$F_{13}(x_1, x_3) = \int \prod_{j=s+1}^{2r} \left(h(x_1, y_j)\, h(x_3, y_j)\, g_j(y_j)\right) \mathrm{d}\nu^{2r-s},$$

$$F_{23}(x_2, x_3) = \int \prod_{j=2r+1}^{2r+q} \left(h(x_2, y_j)\, h(x_3, y_j)\, g_j(y_j)\right) \mathrm{d}\nu^q.$$

By Hölder's inequality,

$$F_{12}(x_1, x_2) \geq \left(\int h(x_1, y)\, h(x_2, y)\, g_{12}(y) \mathrm{d}\nu\right)^s,$$

$$F_{13}(x_1, x_3) \geq \left(\int h(x_1, y)\, h(x_3, y)\, g_{13}(y) \mathrm{d}\nu\right)^{2r-s},$$

$$\left(F_{12}(x_1, x_2)\, F_{13}(x_1, x_2)\right)^{1/2} \geq F_0(x_1, x_2),$$

where

$$F_0(x_1,x_2) = \left(\int h(x_1,y)\, h(x_2,y)\, g_0(y)\mathrm{d}\nu\right)^{\top},$$

$$g_{12}(y) = \left(\prod_{j=1}^{s} g_j(y)\right)^{1/s},$$

$$g_{13}(y) = \left(\prod_{j=s+1}^{2r} g_j(y)\right)^{1/(2r-s)},$$

$$g_0(y) = \left(\prod_{j=1}^{2r} g_j(y)\right)^{1/(2r)}.$$

Denote by I the configuration product of G and estimate

$$\begin{aligned}
I =& I^{1/2} I^{1/2} \\
=& \left(\int F_{12}(x_1,x_2)\, F_{13}(x_1,x_3)\, F_{23}(x_2,x_3)\, f_1(x_1)\, f_2(x_2)\, f_3(x_3)\, \mathrm{d}\mu^3\right)^{1/2} \\
&\times \left(\int F_{13}(x_1,x_2)\, F_{12}(x_1,x_3)\, F_{23}(x_2,x_3)\, f_1(x_1)\, f_3(x_2)\, f_2(x_3)\, \mathrm{d}\mu^3\right)^{1/2} \\
\geq& \int \left(F_{12}(x_1,x_2)\, F_{13}(x_1,x_2)\right)^{1/2} \left(F_{12}(x_1,x_3)\, F_{13}(x_1,x_3)\right)^{1/2} \\
&\times F_{23}(x_2,x_3)\, f_1(x_1)\, f_0(x_2)\, f_0(x_3)\, \mathrm{d}\mu^3 \\
\geq& \int F_0(x_1,x_2)\, F_0(x_1,x_3)\, F_{23}(x_2,x_3)\, f_1(x_1)\, f_0(x_2)\, f_0(x_3)\, \mathrm{d}\mu^3,
\end{aligned}$$

where

$$f_0(x) = \left(f_2(x) f_3(x)\right)^{1/2}.$$

Note that the expression under the last integral is the configuration product of the graph $T_{r,r,q}$ from Example 2, where the vertices u_1, u_2, u_3 correspond to the functions f_1, f_0, f_0, respectively; g_0 is the corresponding function for each of the vertices $w_1, \ldots, w_{2r}$ adjacent to the vertex u_1; the vertices $w_{2r+1}, \ldots, w_{2r+q}$ correspond to the functions $g_{2r+1}, \ldots, g_{2r+q}$, respectively. Since $T_{r,r,q}$ belongs to the class $\mathcal{F}$, inequality (3.2) holds for $T_{r,r,q}$. The second and the third factors in the left-hand side as well as the right-hand side of (3.2) are identical for $T_{r,r,q}$ and G. We have shown that the first factor in the left-hand side for the graph G is greater than or equal to the same expression for $T_{r,r,q}$. Therefore, G satisfies (3.2). □

Proof. Theorem 4.11 Note that the case $v_2(G_1) = v_2(G_2) = 0, e(G_1) = e(G_2) = 0$ is admissible in the proof of Theorem 4.12. Indeed, for such a graph G_2, the used inequality (3.3) is valid. Therefore, Theorem 4.11 is a specific case of Theorem 4.12. □

Proof. Theorem 4.13 Using induction on the number of edges of G, it is easy to check that conditions (a) and (b) are sufficient. Let us check that the equality in (3.1) with $v_2(G) > 1$ implies (a). Indeed, according to Theorems 4.8 and 4.9 , inequality (3.5) holds for G, and its right-hand side is

$$\left(\int (\varphi(x))^{e(G)/v_1(G)} \mathrm{d}\mu\right)^{v_1(G)} \nu(Y)^{v_2(G)-e(G)}.$$

The left-hand sides of (3.1) and (3.5) are identical, and the right-hand side of (3.1) is

$$\left(\int \varphi(x) \mathrm{d}\mu\right)^{e(G)} \mu(X)^{v_1(G)-e(G)} \nu(Y)^{v_2(G)-e(G)}.$$

Since $e(G) = v_1(G) + v_2(G) - 1 > v_1(G)$, the equality

$$\left(\int \varphi(x)^{e(G)/v_1(G)} \mathrm{d}\mu\right)^{v_1(G)} = \left(\int \varphi(x) \mathrm{d}\mu\right)^{e(G)} \mu(X)^{v_1(G)-e(G)}$$

is attained only if φ is equal to a constant almost everywhere vith respect to the measure μ. By analogy, if $v_2(G) > 1$, the equality in (3.1) yields (b). □

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