Ramanujan Graphs

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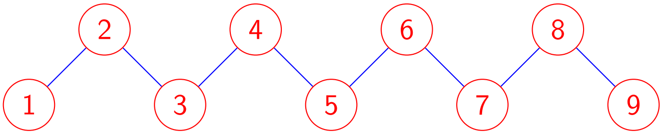
Expander Graphs

Expander graphs are have two notable properties: that they are sparse and highly connected.

Definition

A **sparse graph** contains close to the minimum number of edges possible for a given set of n vertices. In a connected, undirected graph, the minimum number of edges is equal to $n - 1$. In a CS context, even as the total number of vertices approaches infinity, the number of edges in a sparse can be generated in $O(n)$ time.



Expander Graphs

Definition (Edge Boundary)

An **edge boundary**, denoted ∂S, of a set of vertices S is the set of edges attached to both a vertex in S and a vertex in $\overline{S}$.

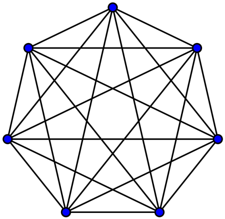
Definition (Cheeger constant)

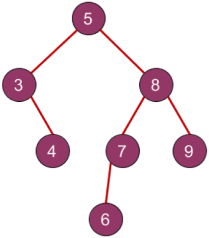
A highly connected graph can be defined in multiple contexts, but the most common one is a bounded **Cheeger constant**, which is denoted:

$$h(G) = \min_{\{S \mid |S| \leq \frac{n}{2}\}} \frac{|\partial S|}{|S|}.$$

This is also a measure of how easily it is (the minimum number of edges it takes) to sever a graph into two pieces, and, in an expander graph, it is bounded to be at least a certain constant.

Expander Graphs





As we can see, expander graphs are difficult to explicitly construct as the number of vertices increases due to how the goals of sparcity and connectivity are at odds with each other. This motivates further study on them in the realm of spectral graph theory.

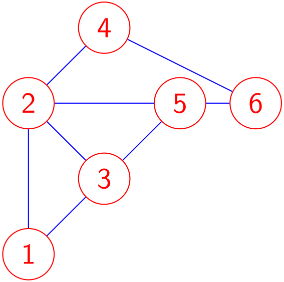
Remark

Expander graphs have many notable applications, such as in data and road networks.

Properties of Adjacency Matrices

Definition

The **adjacency matrix** of a graph has the graph's vertices as rows and columns. If there is an edge connecting vertex i to vertex j, then the element at row i and column j will be labelled with a 1. Otherwise, the elements will be labelled with a 0.



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Properties of Adjacency Matrices

Definition

An **eigenvector** of a matrix is a vector whose direction is unchanged after a matrix transformation.

Definition

An **eigenvalue** of a matrix is the scalar that gets multiplied to an eigenvector during a matrix transformation.

Let $\mathbf{v}$ be an eigenvector and λ be an eigenvalue. Then

$$A\mathbf{v} = \lambda\mathbf{v}$$

It turns out that the first eigenvalue of an adjacency matrix of a d regular graph is d, with a corresponding eigenvector of all ones.

Properties of Adjacency Matrices

Definition

The **spectrum** of a matrix is the set of eigenvalues of that matrix, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \lambda_n$.

Definition

A **spectral gap** is defined to be the quantity $d - \lambda_2$, where d is the degree of a graph and λ_2 is the second eigenvalue.

Properties of Adjacency Matrices

Now I will define a useful relation between eigenvalues and the Cheeger constant:

Theorem (Cheeger Inequalities)

Given a d-regular graph,

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

From this, we can see that a large spectral gap implies a good expander, and a good expander implies a large spectral gap.

Bound on Spectral Gap

This begs the question of how large the spectral gap can actually get, which means we must investigate the upper bound on λ_2

Theorem (Alon-Boppana Bound Simplified)

For d-regular graph G with adjacency matrix A with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$,

$$\lambda_2 \geq 2\sqrt{d-1} - o(1).$$

For our proof, we will use a trace argument to upper bound the trace of an adjacency matrix raised to the $2k$ power to that of the infinite d-regular tree. We then utilize the Catalan numbers and approximation to get our bound.

Proof of Alon-Boppana Bound

First, we can utilize the trace of the matrix, or the sum of the diagonal entries. If we define the characteristic polynomial of an adjacency matrix to be $p(t)$, we have:

$$p(t) = det(A - \lambda I) = (-1)^n(\lambda^n - (Tr(A))\lambda^{n-1} + \ldots + (-1)^n detA)$$
$$p(t) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$$

By Vieta's formulas, we can deduce that the trace is the sum of the eigenvalues, the roots of the characteristic polynomial of the adjacency matrix. Moreover, for a λ_i and its eigenvector $\mathbf{v}$, we have

$$A^k\mathbf{v} = A^{k-1}(A\mathbf{v}) = A^{k-1}\lambda_i\mathbf{v} = \lambda_i^k\mathbf{v},$$

which brings up the useful relation that the eigenvalues of A raised to the kth power are the eigenvalues of A^k.

Proof of Alon-Boppana Bound

From here, we define $\lambda = max(|\lambda_2|, |\lambda_n|)$, where $n \geq 3$. Then, we get the following:

$$Tr(A^{2k}) = \sum_{i=1}^{n} \lambda_i^{2k} \leq d^{2k} + (n-1)\lambda^{2k},$$

where we substituted d for λ_1 and λ for the other eigenvalues.

Lemma (Number of Closed Paths)

$Tr(A^k)$ = sum of diagonal entries of A^k = number of closed paths of length k in A.

We can prove this lemma by the principal of mathematical induction.

Proof of Alon-Boppana Bound

Proof.

By definition, A_{ij} is the number of paths of length 1 from node i to node j. Assume that A_{ij}^{b-1} is the number of paths of length $b-1$ from node i to node j. Then,

$$A^{b-1} \cdot A = A^b$$

$$A_{ij}^b = A_{i1}^{b-1}A_{1j} + A_{i2}^{b-1}A_{2j} + \ldots + A_{in}^{b-1}A_{nj} = \sum_{x=1}^{n} A_{ix}^{b-1}A_{xj}.$$

Each individual product is equivalent to the total number of paths of length b that go to node x right before going to node j, and summing them up gives us the number of paths from i to j of length b. By the principle of mathematical induction, $A_{ij}^k =$ the number of paths of length k from i to j. Hence, the sum of diagonal entries would be the sum of the number of closed paths. ■

Proof of Alon-Boppana Bound

- $Tr(A^{2k}) = (\#$ of closed paths of length $2k$ in $G) \geq n \cdot (\#$ closed paths of length $2k$ in an infinite d regular tree).
- Number of possible closed paths in infinite d-regular tree: Dyck path (can only increase or decrease the distance to the starting node by 1)

Therefore, the number of paths would correspond to the kth Catalan number. From each choice, we have a minimum of $d-1$ edges to pick from (as the tree is d-regular). Hence, we get

$$d^{2k} + (n-1)\lambda^{2k} \geq n \cdot \frac{1}{k+1}\binom{2k}{k}(d-1)^k.$$

Proof of Alon-Boppana Bound

Moving around the terms and dividing by $n-1$ from both sides, we get

$$\lambda^{2k} \geq \frac{n}{n-1}\left(\frac{1}{k+1}\binom{2k}{k}(d-1)^k\right) - \frac{d^{2k}}{n-1}.$$

After taking the $\frac{1}{2k}$th power of the inequality and some approximations, we get

$$\lambda \geq 2\sqrt{d-1} - o(1).$$

Ramanujan Graphs

Definition

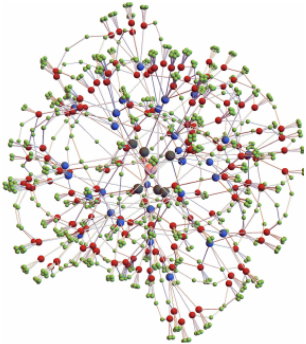
A **Ramanujan graph** with spectrum $d = \lambda_1 \geq \lambda_2 \geq ...\lambda_n$ has $\lambda_2 \leq 2\sqrt{d-1}$. This relationship implies that Ramanujan graphs have small second eigenvalues and, when coupled with the Alon-Boppana Bound, makes them incredibly good expander graphs.

Currently, there are only a couple of explicit constructions of Ramanujan Graphs, so Ramanujan Graphs are still an area of developing research.

Ramanujan Graphs

Theorem (Lubotzky-Phillips-Sarnak)

For every prime number p, infinite sequences of Ramanujan Graphs exist for $d = p + 1$.



LPS Construction

Definition (Generating Set)

A **generating set** of a group is a subset of the group such that any element within a group can be constructed through the operation defined by the group applied to elements of the generating set and their inverses.

Definition (Cayley Graph)

A **Cayley Graph** $\Gamma = \Gamma(G, S)$ is constructed by:

- A vertex set G
- Each edge $s \in S$ assigned a color s belonging to the generating set.
- There is an edge of color s connecting g and gs, where gs represents the operation of the group applied between g and s.

LPS Construction

The Construction:
Let p and q be two distinct primes, both $1 \pmod 4$ and i an integer such that $i^2 \equiv -1 \pmod p$. There are a total of $8(q+1)$ integer solutions to $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = q$. There are $q+1$ solutions such that $\alpha_0 > 1$ and is odd, and the others are even. Associate with each set of solutions the following:

$$\alpha = \begin{bmatrix} \alpha_0 + i\alpha_1 & \alpha_2 + i\alpha_3 \\ -\alpha_2 + i\alpha_3 & \alpha_0 - i\alpha_1 \end{bmatrix}$$

Taking this set of matrices as the generating set S of a Cayley graph, there is both a bipartite and a non-bipartite construction of a $q+1$ regular Ramanujan graph.

Thanks for listening.