Hackenbush

An Introduction to Combinatorial Game Theory

Ethan Chen

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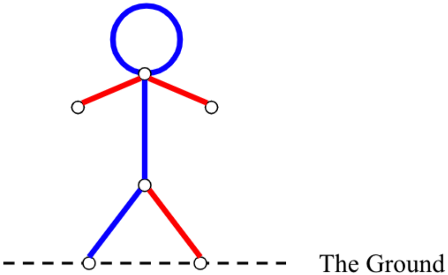
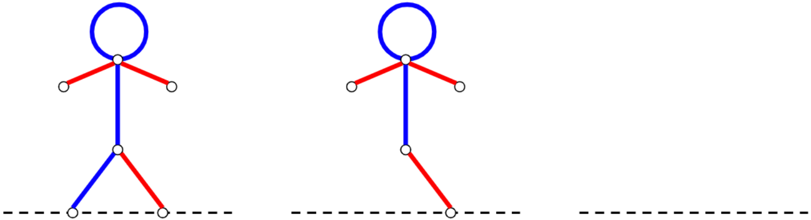
Rules of Blue Red Hackenbush

- Left and Right takes turns moving
- Left cuts bLue, Right cuts Red
- Any edge not connected to the ground disappears
- First person to run out of moves loses.



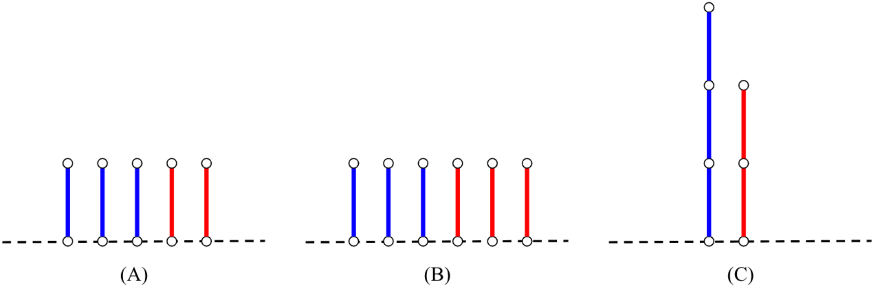
Sample Game



(A) (B) (C)

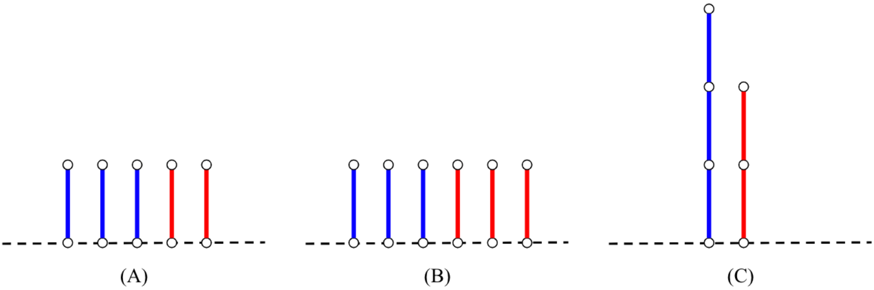
Evaluating Games

- No edges affect each other
- Blue edge $= +1$
- Red edge $= -1$



Zero Game

- Whoever moves first loses
- Position (B) is a zero game

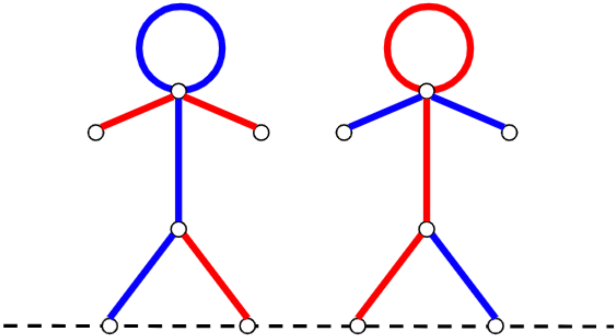


Sum of Games

- Combine two positions which aren't connected by any edge
- Value of new position = sum of the value of old positions

Negative of a game

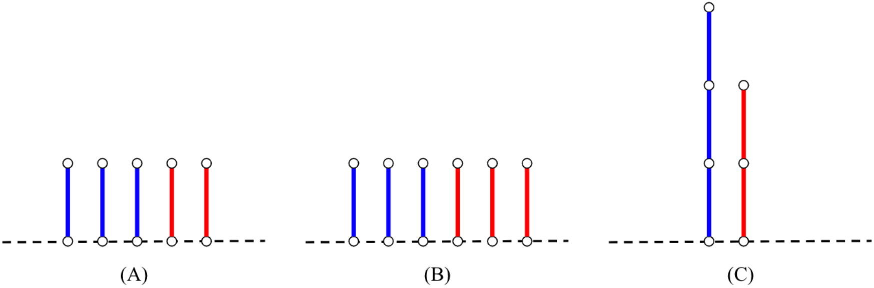
The sum of a game and its negative is a zero game:



How to express a game using mathematical notation

- $Notation\{L|R\}$
- $L =$ the set of all games after Left's move
- $R =$ the set of all games after Right's move
- Can be simplified to $\{a|b\}$ where a, b are the most favorable positions for each player respectively.

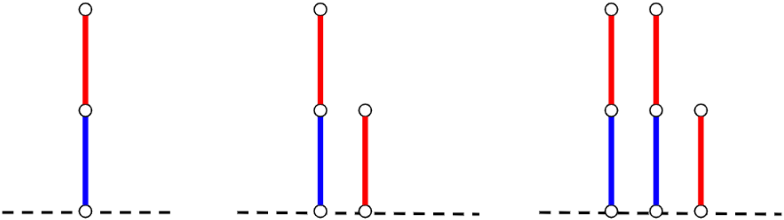
How to express a game using mathematical notation



Position C can be expressed as $\{-2, -1, 0|2, 3\}$ which simplifies to $\{0|2\}$

Fractional Values

Advantage does not need to be an integer!



(A) (B) (C)

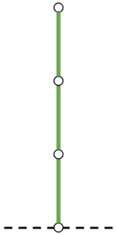
Winning Strategy

- If a position is positive, Left will win if playing optimally
- If a position is negative, Right will win if playing optimally

Therefore, to win Hackenbush, you can evaluate the position of every possible game after your move and choose the game which benefits you the most. (but this is easier said than done)

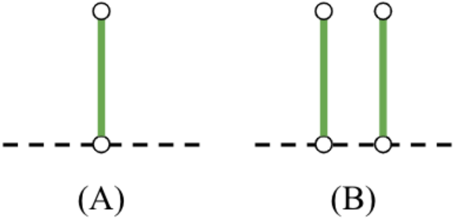
Rules of Green Hackenbush

- All edges are green
- Either player can cut a green edge
- Impartial game



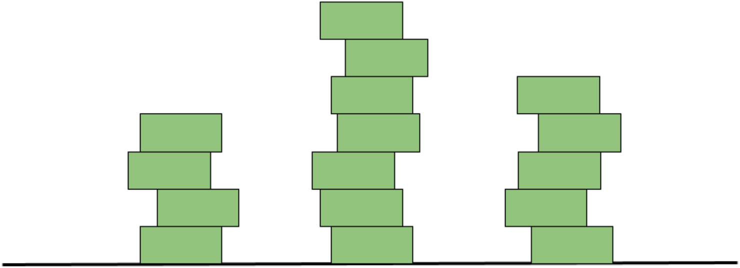
A New Type of Game

- Position A is a fuzzy game, denoted as $\{0|0\} = *$
- First to move wins
- Position B is a zero game, thus $* + * = 0$



The Game of Nim

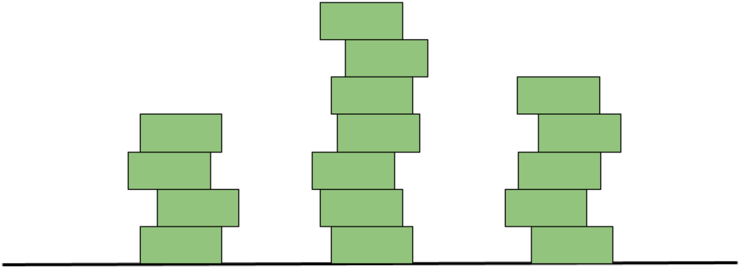
- In Nim, there are several heaps of arbitrarily many objects
- Left and Right takes turns choosing a heap and removing as many objects as they want from that heap
- A heap of size n is denoted as a game with value $*n$, called a "nimber".



Nim Addition Rule

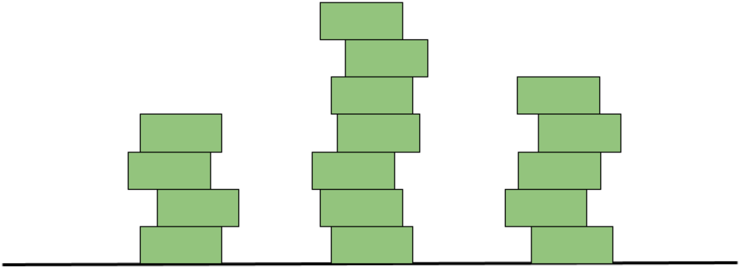
Unlike Blue Red Hackenbush, Nim positions are represented with nimbers, which do not follow the regular addition rules for numbers. Instead, adding nimbers involves taking their bitwise XOR.

The following position has value $*4 + *7 + *5 = *(4 \oplus 7 \oplus 5) = *6$



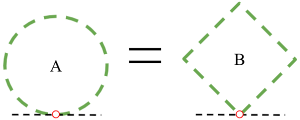
Winning Strategy in Nim

To win at Nim, you must reduce the current position to a zero position. For example, in this position, the first player can win by reducing the heap of size 7 to 1, causing the new position to have the value
$*4 + *1 + *5 = *(4 \oplus 1 \oplus 5) = *0 = 0$

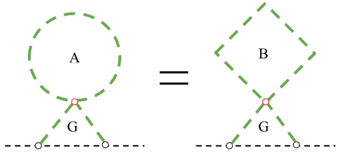


Colon Principle

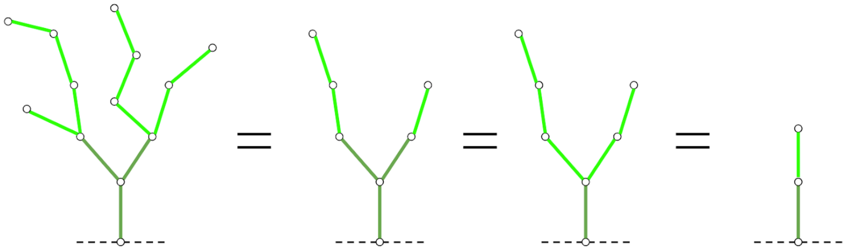
If



Then

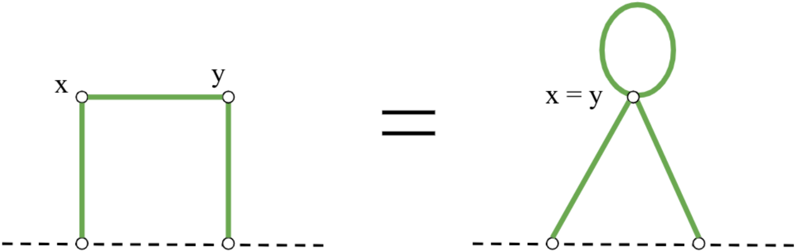


Colon Principle Example

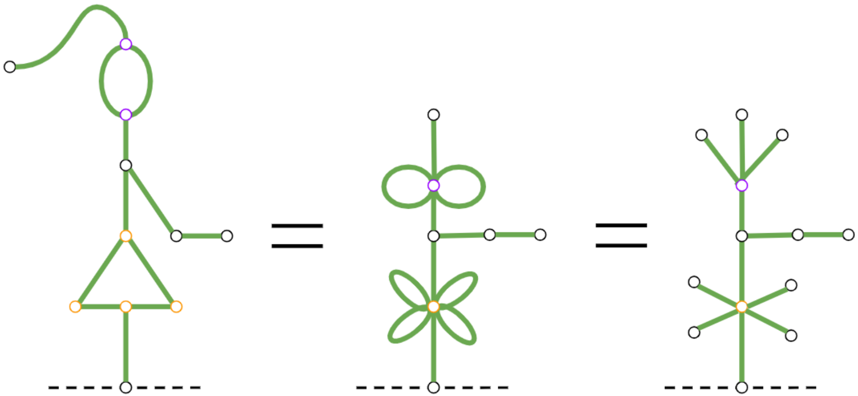


Fusion Principle

You can fuse any number of nodes part of the same cycle and retain the same value.



Fusion Principle Example



Winning Strategy

Much like Nim, the winning strategy in Green Hackenbush is to reduce the position to a zero position for your opponent to move. A simple brute force method would be to evaluate every possible position and move to a zero position.

Thank you for listening