# Hackenbush Game Theory

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June 2024

**Abstract**

Combinatorial game theory is the study of certain types of mathematical games. In this paper, we will discuss in-depth analysis on the game of Hackenbush, such as evaluations of game positions, winning strategies, etc.

## 1 Introduction

A combinatorial game typically follows these restrictions:

1. There are two players who take turns making moves

When there are more than two players, the game becomes very difficult to analyze. For example, if Player 3 is certain that they will lose, would they make moves that would sabotage Player 1 or Player 2?

2. There are several possible positions that can be obtained through a series of moves from a starting position

In a game of chess, for example, there is a starting position on the board before either player makes a move. When the game starts, either player alternates making moves to reach new positions.

3. The game has complete information, or in other words, each player knows exactly what is happening

A game such as Battleships is an example of a game without complete information. At some point in the game, Player 1 does not know the exact locations of their opponent's ships. (which is what makes the game fun!)

4. There are no chance moves

The rolling of the die, shuffling of a deck, are all considered chance moves, because the outcome is uncertain. [BCG82]

In this paper, we will be discussing the combinatorial game Hackenbush invented by John Horton Conway. It is played between two players, Left and Right. The game involves a graph consisting of nodes and edges, and a line in which some nodes may lie on called "the ground".

## 2 Blue Red Hackenbush

The most basic form of Hackenbush is Blue Red Hackenbush, played on a graph of blue and red edges. (See Figure 1)

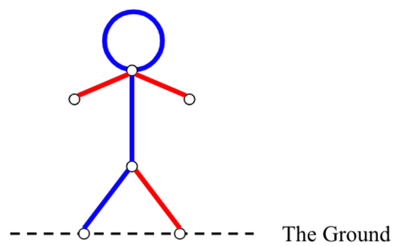


Figure 1: A Hackenbush Position

A player starts by erasing an edge of their color ("L" is for bLue and "R" is for Red) along with any edge that is no longer connected to the ground. An edge is said to not be connected to the ground if you cannot travel from that edge to the ground by following a path of other edges. Each player takes turns erasing edges in this fashion until one player is unable to move because there are no more edges of their color, in which case they lose.

The following is an example of a game played by Left and Right with Figure 1 as the starting position:

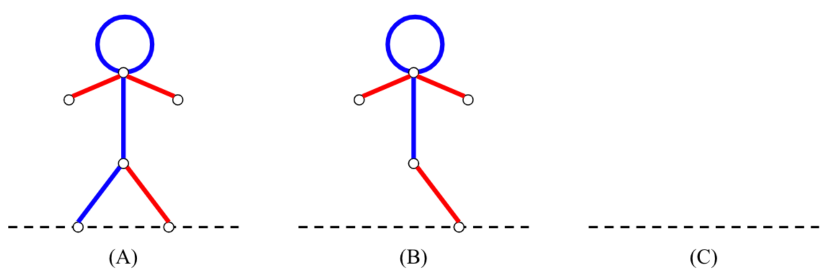


Figure 2: A game of Hackenbush

Left starts by cutting the stick figure's left leg, going from (2A) to (2B). Right then cuts the stick figure's other leg, along with the rest of the stick figure which is no longer connected to the ground. When it is Left's turn, there are no more blue edges, and thus, Left loses.

Since Right won, one might be tempted to say that this position is in favor of Right. However, after a second thought, it is quite obvious that Left did not play optimally here. If Left instead started by cutting the stick figure's head or body, Left would have won. [BCG82]

And also, what happens if Right starts?

# 3 How to evaluate simple Hackenbush positions

The position in Figure 1 is quite difficult to evaluate, so let's look at a simpler example.

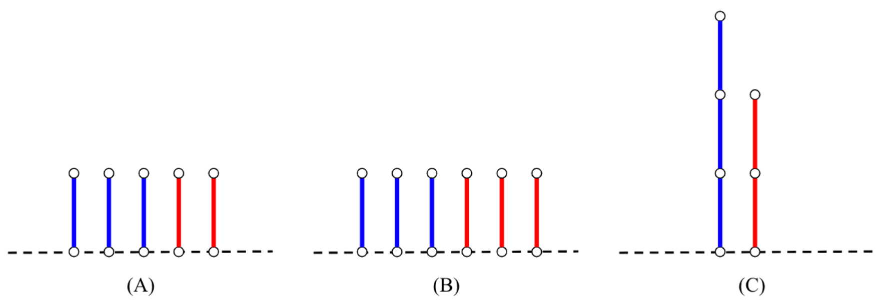


Figure 3: Simple Hackenbush Positions

## 3.1 The value of a position

In Figure 3A, Left has a full edge advantage on Right. No matter who starts, Left will always win. We can say that in this position, Left has $3 - 2 = 1$ "free move(s)," so this position has a value of 1. Similarly, in Figure 3C, Left also has a full edge advantage on Right. As long as Left plays sensibly, this position is equivalent to that of Figure 3A, so it also has a value of 1.

If we add a free move for Right to obtain Figure 3B, Left would have $3-3 = 0$ spare moves. It turns out, in this position, whoever is the first to move is the loser. This position is commonly referred to as a "zero position" or a "zero game."

If a game's value is positive, Left will win no matter who starts, if the game's value is 0, the second player to move wins, and if the game's value is negative, Right wins. [BCG82]

## 3.2 The $\{a, b, c, \cdots \mid d, e, f, \cdots\}$ notation

We can represent a game in the form $\{a, b, c, \cdots \mid d, e, f, \cdots\}$ where $a, b, c, \cdots$ are the values of the position after each possible left move, and $d, e, f, \cdots$ are the values of the position after each possible right move. For example, Figure 3C can be written as $\{-2, -1, 0 \mid 2, 3\}$. If Left removed the bottom-most blue edge, the value would be $0 - 2 = -2$; if they removed the middle blue edge, the value would be $1 - 2 = -1$; if they removed the top-most blue edge, the value would be $2 - 2 = 0$. Similarly, if Right removed the lower red edge, the value would be $3 - 0 = 3$, and if they removed the upper red edge, the value would be $3 - 1 = 2$.

Since the greater the value of a game is, the more favorable it is for Left, the best option for Left is the move which would result in the greatest value, and similarly, the best option for Right is the move which would result in the least value. Thus, we can say $\{-2, -1, 0 \mid 2, 3\} = \{0 \mid 2\}$. We know that the value of Figure 3C is 1, so we can say $\{0 \mid 2\} = 1$.

In general, we can often say that $\{a, b, c, \cdots \mid d, e, f, \cdots\} = \{x \mid y\}$, where $x$ is the most favorable outcome for Left, and $y$ is the most favorable outcome for Right. [BCG82]

## 3.3 The sum of two games

We can combine multiple games to get a new game which is the sum of the two components, simply by allowing either player to move from either component during their turn. For example, we can combine the positions in Figure 3A and Figure 3B to get a game with 6 free moves for Left and 5 free moves for Right. The value of such a game would simply be the sum of the values of either component. [BCG82]

We can even add different combinatorial games, such as a Hackenbush position plus a Nim position!

## 3.4 The negative of a game

For a game, $G$, the negative of the game, $-G$, can be added to itself to get a zero game. Observe the following position:

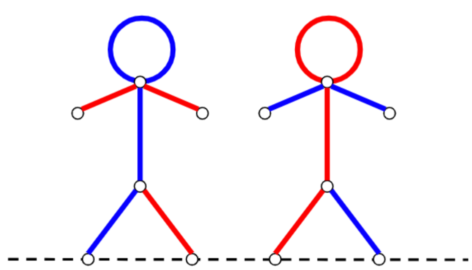

Figure 4: Stick Figure and his friend Erugif Kcits

Although we don't know the value of one stick figure, we can see that this position is a zero position, as the second player to move can win by copying the first player's move in the corresponding stick figure.

Notice that all we did was interchange the color of each edge to produce the negative of the stick figure. In general, for a game $G = \{a, b, c, \cdots \mid d, e, f, \cdots\}$, the negative of the game is $-G = \{-d, -e, -f, \cdots \mid -a, -b, -c, \cdots\}$.

# 4 Fractional values in Hackenbush

If you have played chess before, you might be familiar with non-integer evaluations. If so, you might not be surprised to learn that Hackenbush also has fractional advantage values.

## 4.1 Half a move!?

Take a look at Figure 5A:

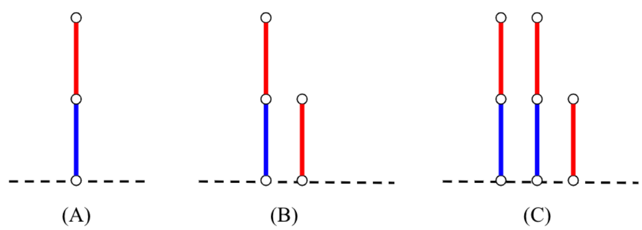


Figure 5: More Hackenbush Positions [Bar06]

If Left starts, they win after deleting the bottom edge and forcing Right to move in an empty position. If Right starts, Left still wins after each player moves. So this position has a positive value, even though there are the same number of blue and red edges. Is it necessarily a 1-move advantage for Left?

We can test this by adding a full move advantage for Right to get Figure 5B. Now, if Left starts and deletes their only edge, Right can delete their own remaining edge and Left loses. If Right starts, they win after deleting the edge above Left's edge and forcing Left to move first in a zero game. So Left's advantage in Figure 5A is between 0 and 1.

Can we reasonably assume it is 1/2 then? We can test this by adding Figure 5A and 5B to get 5C. Theoretically, Figure 5C should be a zero game, because $1/2 + 1/2 - 1 = 0$.

If Left starts, any move turns the position into Figure 5B, which we know Right wins. If Right starts, they have two options: to choose an edge above a blue edge, or the edge on the ground. If they choose the former, Left can delete the other blue edge with a red edge above and the position becomes a zero game with Right to move. If they choose the latter, Left can turn the position into Figure 5A, which we know Left wins. Since in this position, the second player to move wins, this is a zero game, so our original hypothesis is correct.

Notice that the game in Figure 5A can be written as $\{0 \mid 1\}$. We have just shown that $\{0 \mid 1\} = 1/2$.

In a similar way, you can prove that the following Figure 6A has value 1/4, and Figure 6B has value 1/8. [Dav11]

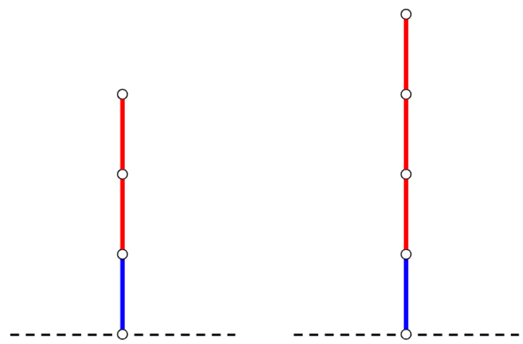


Figure 6: Fractional Hackenbush

### 4.2 Fractional positions with denominators of powers of two

What happens if we combine the games 1/4 and 1/8 (that is, the positions of Figure 6A and 6B)? We get the game of $1/4 + 1/8 = 3/8$. We can write either game as $\{0 \mid 1/2\}$ and $\{0 \mid 1/4\}$ respectively, and our game is the sum $\{0 \mid 1/2\} + \{0 \mid 1/4\}$. Left's options after they move are the positions $0 + \{0 \mid 1/4\} = 1/8$ if they move in the first component, or $\{0 \mid 1/2\} + 0 = 1/4$ if they move in the second component. Since $1/4 > 1/8$, the latter is obviously the better move. In a similar fashion, you can show that Right's best move is in the first component, with a resulting value of 1/2. Thus, we have shown that $\{1/4 \mid 1/2\} = 3/8$. [BCG82]

In precisely the same way, we can generalize, and say that

$$\left\{ \frac{p}{2^n} \mid \frac{p+1}{2^n} \right\} = \frac{2p+1}{2(n+1)}$$

## 5 The Simplicity Rule

What is the value of $\{1/4 \mid 1\}$? You may be tempted to just take the average and say it's 5/8, but is that necessarily the case?

We can check that by seeing whether the following sum is a zero game:

$$\{1/4 \mid 1\} + \{-3/4 \mid -1/2\} = X + (-5/8)$$

We know this sum is a zero game if neither player has a move that puts them in a winning position. If it is Left to move and they choose to move from the component $X$, then the resulting game is $1/4+(-5/8) = -3/8$, and if it is Right to move and they choose to move from the component $X$, then the resulting game is $1+(-5/8) = 3/8$. However, Right can choose to move in the component $-5/8$, leading to the position

$$X + (-1/2) = \{1/4 \mid 1\} + \{-1 \mid 0\}$$

Now, it is Left to move, and they still cannot move in the component $X$ for the same reason as before, so they must move in the other component, leading to the position

$$X + (-1) = \{1/4 \mid 1\} + (-1)$$

Once again, it is Right to move, and they have a winning move in component $X$, which results in a zero game with Left to move first. Since Right has a winning strategy in this game, we have disproved that $X = -5/8$.

To actually calculate the value of $\{1/4 \mid 1\}$, we can use the simplicity rule.

### 5.1 What is the Simplicity Rule?

The simplicity rule states that the value of a game $\{a \mid b\}$ is the simplest number strictly between $a$ and $b$. A number is said to be simpler than another if it has a lesser denominator, or if the denominators are the same, a lesser magnitude.

### 5.2 Proof

Let's say we have a game $X = \{a \mid b\}$ and a game $Y = \{c \mid d\}$, where $c \geq a$ and $d \geq b$. Observe the game $\{a \mid b\} + (-\{c \mid d\}) = \{a \mid b\} + \{-d \mid -c\}$. Left's options are $a + \{-d \mid -c\}$ which Right can respond with $a - c$, or $\{a \mid b\} - d$, which Right can respond with $b - d$. Both $a - c$ and $b - d$ are winning positions for Right. Similarly, if Right starts, Left can respond with $b - d$ or $a - c$, which are both winning positions for Left. Therefore, $\{a \mid b\} + (-\{c \mid d\}) = 0$, so $\{a \mid b\} = \{c \mid d\}$. We can use this argument to show that there are positions $\{c \mid d\}$ which yield the simplest number strictly between $\{a \mid b\}$ which have the same value $\{a \mid b\}$. [BCG82]

Using the simplicity rule, we see that $\{1/4 \mid 1\} = \{0 \mid 1\} = 1/2$.

The simplicity rule is used very frequently throughout combinatorial game theory to show the value of a game.

## 6 Green Hackenbush

Green Hackenbush is a variation of Hackenbush which has essentially the same rules as Blue-Red Hackenbush, except the graph is composed entirely of green edges. Either player is allowed to cut a green edge.

Green Hackenbush is an example of an impartial game. An impartial game is where either player has access to the same moves, and the only difference between the two players is who gets to move first.


Figure 7: Green Hackenbush Positions

Green Hackenbush introduces us to a new type of game. We know that when a game's position is positive, it is a winning position for Left, when it is negative, it is a winning position for Right, and when it is 0, the second player to move wins. Observe Figure 7. Notice that the first player to start can simply remove the only edge, and thus win the game. So this position is not a winning position for Left or Right, but it is not a zero game either, because the second player to move loses.

## 6.1 The value of a star

We can write the game as $\{0 \mid 0\}$, but we are unable to use the simplicity rule as there is no number strictly between 0 and 0. However, we can safely assume this value is less than $\{0 \mid 1\}$, $\{0 \mid 1/2\}$, $\{0 \mid 1/4\} \ldots$ Similarly, we can assume this value is greater than $\{-1 \mid 0\}$, $\{-1/2 \mid 0\}$, $\{-1/4 \mid 0\} \ldots$ Although this value seems to approach 0, it is important to remember that it is not precisely 0.

Since this type of game is so special, it is denoted with a special value, namely, the $*$ (star). A game whose value is $*$ is said to be a fuzzy game.

In Figure 7B, we have combined two of Figure 7A to get the game $* + *$. In this game, the first person to start can cut one of the two edges, in which the second person can respond by cutting the other edge and winning the game. Therefore, $* + * = 0$.

## 6.2 The value of $x$ star

If we combine the games 3/4 and $*$, we have $\{1/2 \mid 1\} + \{0 \mid 0\}$. Left's options are $1/2 + *$ and $3/4 + 0$, and Right's options are $1 + *$ and $3/4 + 0$. Since the value of $*$ is less than 3/4, Left and Right's best options are both 3/4. This shows that $\{3/4 \mid 3/4\} = 3/4 + *$, or more generally, $\{x \mid x\} = x + *$ for some number $x$.

# 7 Nim and Nimbers

What if we had a position consisting of a long stem of green edges?

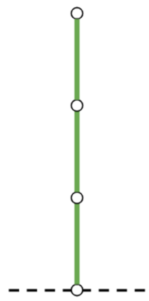

Figure 8: A green stem

To evaluate this, we must observe Nim.

## 7.1 The game of Nim

In Nim, positions consist of several stacks of arbitrarily many objects. Left and Right take turns moving, and during a turn, the player can choose a stack to remove any number of objects from. The player who removes the last object wins.

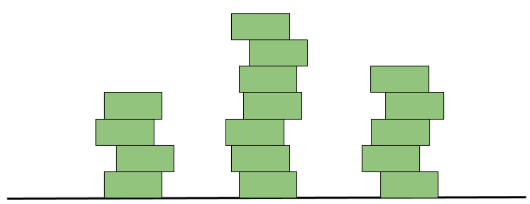

Figure 9: A Nim position

Let's denote a nim heap of size $n$ to have the value $*n$. Note that this is not the same as $n + *$, but rather it is a special kind of number, known as a *nimber*.

Here are some basic properties of nimbers: [BCG82]

$$*0 = \{|\} = 0$$
$$*1 = \{*0 \mid *0\} = \{0 \mid 0\} = *$$
$$*2 = \{*0, *1 \mid *0, *1\}$$
$$*3 = \{*0, *1, *2 \mid *0, *1, *2\}, \ldots$$
$$*n = \{*0, *1, \ldots, *(n-1) \mid *0, *1, *2, \ldots, *(n-1)\}$$

We also know that adding two nim heaps of the same size is a zero game using a similar argument as in Section 3.4.

## 7.2 The Mex Rule

The Mex rule states that for any impartial game $G = \{*a_0, *a_1, *a_2, \ldots \mid *a_0, *a_1, *a_2, \ldots\}$, $G$ has value $*x$, where $x$ is the smallest whole number not equal to any $a_i$.

To prove this, consider the sum of games $G + H + J + \ldots$ (where $H, J, \ldots$ does not necessarily have to be an impartial game). Without loss of generality, suppose Left has a winning strategy in $*x + H + J + \ldots$. Since the game $G$ has all the options that $*x$ has, if his/her strategy requires a move in $*x$, that move will be available. If Right moves in $*x$, this does not interfere with Left's strategy, as Left is already counting $G$ to act as $*x$. If Right instead moves to an option $*a_i$ greater than $*x$, Left can simply move the position back to $*x$.

Thus, $G$ can simply be regarded as a Nim heap of size $x$. [BCG82]

## 7.3 Nimber addition

In Nim, there are often several heaps of objects. To evaluate such a position, we must add nimbers. Adding two nimbers does not simply involve taking the sum of the numerical value of the nimbers, as $*1 + *1$ is a zero game. It turns out, nimber addition is actually as simple as taking the bitwise XOR of the two numbers. For example, $*1 + *3 + *2 = *(1 \oplus 3 \oplus 2) = *0 = 0$. However, this seems too convenient to be true, so let's take a look at the proof.

## 7.4 Nimber addition proof

Before diving into the proof, let's first prove the Parity Rule. The Parity Rule states that

$$*1 + *k = *(k+1) \text{ if } k \text{ is even and } *(k-1) \text{ if } k \text{ is odd.}$$

We will use induction to prove the Parity Rule.

Consider the base case for $k = 0$ (even) and $k = 1$ (odd):

$$*1 + *0 = \{*0 \mid *0\} = *1,$$

$$*1 + *1 = \{*1, *1 \mid *1, *1\} = *0.$$

These can be verified using the Mex Rule. We need to show that if the theorem holds for every $m \in \mathbb{N}$ such that $m \leq n$, then

$$*1 + *(n+1) = *(n+2) \text{ if } (n+1) \text{ is even and } *n \text{ if } (n+1) \text{ is odd.}$$

Suppose $n+1$ is even. Then

$$\begin{aligned} *1 + *(n+1) &= \{*(n+1), *1 + *(n), *1 + *(n-1), \ldots, *1 + *1, *1 + *0 \mid \ldots\} \\ &= \{*(n+1), *(n-1), *(n), *(n-2), \ldots, *0, *1 \mid \ldots\} \\ &= *(n+2) \end{aligned}$$

by the Mex Rule.

Now suppose $n+1$ is odd. Then

$$\begin{aligned} *1 + *(n+1) &= \{*(n+1), *1 + *(n), *1 + *(n-1), \ldots, *1 + *1, *1 + *0 \mid \ldots\} \\ &= \{*(n+1), *(n+1), *(n-2), *(n-1), \ldots, *0, *1 \mid \ldots\} \\ &= *n \end{aligned}$$

by the Mex Rule.

With the Parity Rule established, let's now prove the Nimber Addition Rule. We can accomplish this by first proving that for every $n, k \in \mathbb{N}$ such that $n < 2^k$,

$$*n + *2^k = *(n + 2^k).$$

If this is true, we can write any nimber as the sum of distinct powers of two, and when adding two nimbers, $*x$ and $*y$, we can simply cancel common powers of 2 of both $x$ and $y$, since their sum is a zero game. Note that this is essentially the same thing as taking the bitwise XOR of $x$ and $y$.

We can proceed with induction. Consider the base case when $k = 1$:

$$*1 + *2 = *3.$$

We must show that for some

$$k \in \mathbb{N}, \text{ if for every } n, l \in \mathbb{N} \text{ such that } n < 2^l, l < k,$$

$$*n + *2^l = *(n + 2^l),$$

then for every

$$n \in \mathbb{N} \text{ such that } n < 2^k,$$

$$*n + *2^k = *(n + 2^k).$$

To prove this, we can use induction again. To prove the base case, we can use the Parity Rule to show that

$$*1 + *2^k = *(2^k + 1).$$

For the inductive step, we just need to show that given

$$\forall m \in \mathbb{N}, m \leq n,$$

$$*m + *2^k = *(m + 2^k),$$
$$*(n+1) + *2^k = *(n + 1 + 2^k).$$

If $n$ is even,

$$*(n+1) = *n + *1,$$

so

$$*(n+1) + *2^k = *n + *1 + *2^k = *1 + *(n + 2^k) = *(1 + n + 2^k)$$

by the Parity Rule. A similar strategy can be used to prove this for odd $n$.

This concludes our inductive step, allowing us to show that the sum of two nimbers is just the bitwise XOR of their numerical values. [Bar06]

# 8 Green Hackenbush Solved

You've probably noticed now that a green Hackenbush position consisting of only green stems is just the game of Nim. But what about a more complicated position?

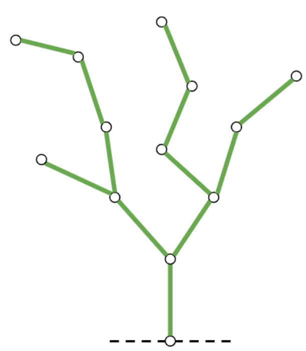

Figure 10: Green Tree

To simplify this position, let's take a look at a new principle.

## 8.1 The Colon Principle

The Colon Principle states that for a game $G$ and two games $A$ and $B$,

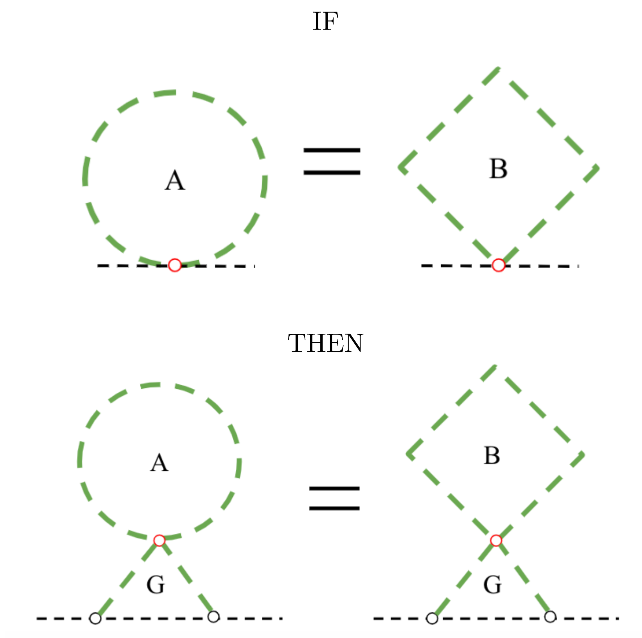


Figure 11: The Colon Principle Visualized

We can prove this by showing $G_x : A + G_x : B = 0$, where $G_x : A$ denotes the game $A$ attached on node $x$ of the game $G$. If the first player moves in $G$, the second player can respond by mirroring that move in the other $G$. If the first player moves in $A$ or $B$, they change the value of that to some $*x$, which the second player can respond by changing the other component to the value $*x$. If this goes on, the second player will never run out of a move, as long as the first player has a move, thus winning the game, proving that $G_x : A + G_x : B$ is indeed a 0 game. [Bar06]

We can use the Colon Principle along with nimber addition to analyze the position in Figure 12.

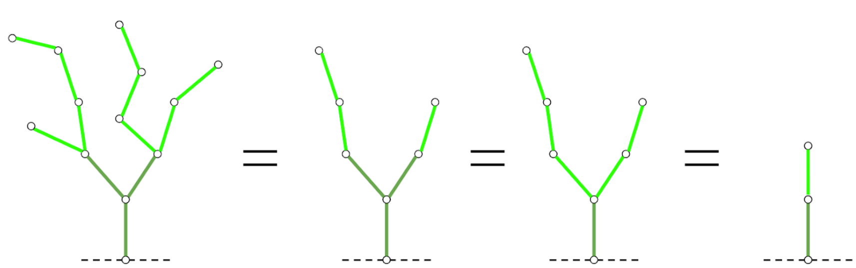

Figure 12: Simplifying a green tree

After simplification, we see that the value of this tree is $*2$.

## 8.2 The Fusion Principle

A cycle in Green Hackenbush is a set of edges that form a loop. The ground can be considered an edge.

The Fusion Principle states that you can fuse any two nodes that are part of the same cycle without changing the position's value. Fusing two nodes is when you bring the two nodes together into one node, and bend any edge that was connecting them into loops.

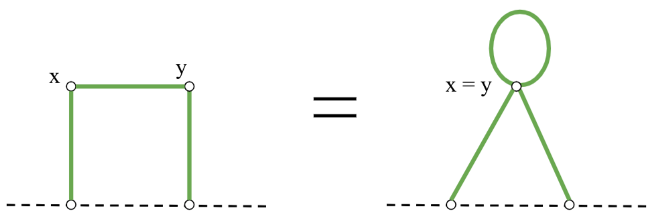


Figure 13: The Fusion Principle visualized [BCG82]

The proof of the Fusion Principle is lengthy and rigorous, and we will not be covering it in this paper. A full length proof can be found in the references below.

Here is an example of finding the value of a game using the fusion principle. Certain nodes are highlighted to make it easier to see which nodes are being fused.

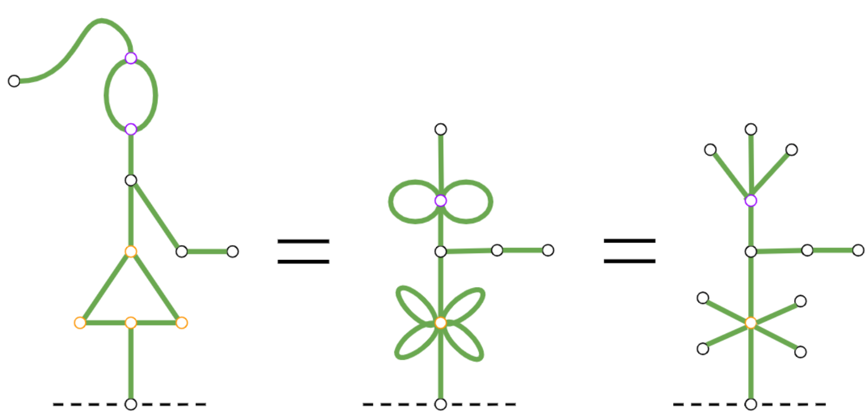

Figure 14: Using the Fusion Principle [BCG82]

We can see that this position has been simplified to a tree, which can be solved as follows using the Colon Principle. Once again, certain edges are highlighted to make it easier to follow.

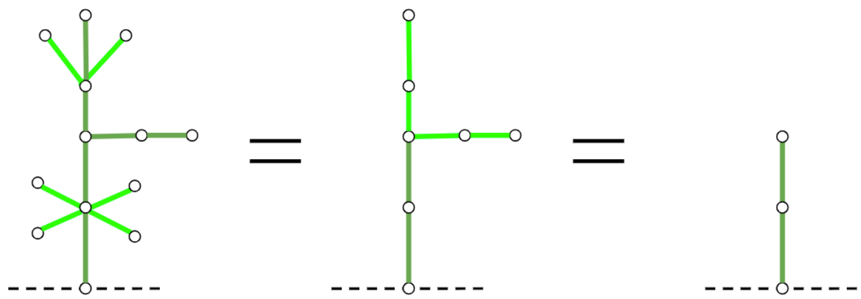

Figure 15: Solved!

The fusion principle completely solves Hackenbush, meaning that you can determine the value of any green Hackenbush position.

# 9 Conclusion

Although we covered a complete winning strategy for Green Hackenbush, there are many techniques which we have not covered that can speed up the process greatly. Additionally, there are many, many more types of positions for Blue Red Hackenbush, as well as a whole new type of Hackenbush called Hackenbush Hotchpotch which is played with edges of all 3 colors, that we have not covered in this paper.

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