Brownian Motion

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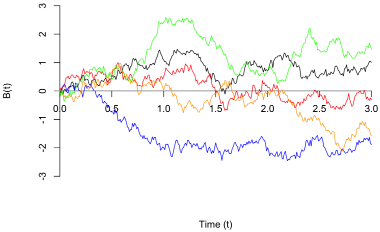
- mathematically formalized by Norbert Wiener $\rightarrow$ Wiener Process

Applications of Brownian Motion



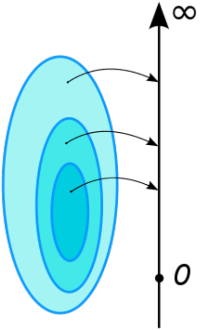
Important in biology(movements of liquids, evolution), chemistry(kinetic theory, stability), physics(heat diffusion, conduction,etc), finance (modeling stock prices), etc

Measure Theory

Definition

Let a measurable space be (X, A, μ)

1. X : set
2. A: special collection of subsets of X
3. μ : a measure, maps the sample space onto $[0, \infty]$, giving it a generalized length/volume



Probability Space

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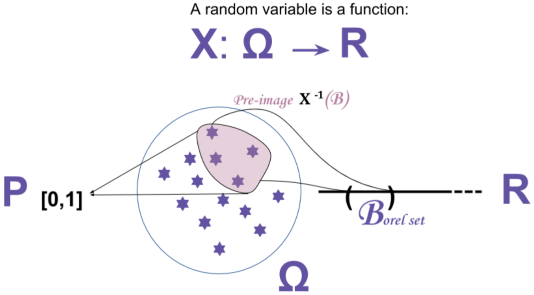
1. Ω: the **sample space** of all possible outcomes.
2. A: a **sigma-algebra**, a collection of subsets of Ω. Each set is called an event.
3. $\mathbb{P}$: **probability measure** that maps events onto their probability values from 0 to 1, with 0 being impossible* and 1 being certain. Because all events are "cut out" of Ω, $\mathbb{P}(\Omega) = 1$.

Random Variables

Definition

A **random variable** on the probability space $(\Omega, A, \mathbb{P})$ is an σ-measurable function from the set of all possible outcomes to the set of real numbers: $X : \Omega \to \mathbb{R}$.



Probability Preliminaries

Expectation

$$\mathbb{E}(X) = \int_{\Omega} X d\mathbb{P} \tag{0.1}$$

The bounds Ω being the set of all functions in Ω from $(-\infty, \infty)$.

Variance

$$Var(X) = E((X - E(X))^2), \tag{0.2}$$

denoted as σ^2.

Covariance

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]. \tag{0.3}$$

Definition

A **normal distribution** or **Gaussian distribution** is a continuous probability for a real valued random variable, typically shown a bell curve. The probability density function is as follows:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{0.4}$$

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Definition

A stochastic process X_t is a collection of random variables indexed by time.

Standard Brownian Motion

Definition

A stochastic process $(B_t)_{t \geq 0}$ is a Standard Brownian Motion if it satisfies the following properties:

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3. **Stationary Increments of Normal Distribution:** Each interval of $B_{(s+t)} - B_s$, given that $s < t$, is normally distributed with expectation 0 and variance s shown by $\sim N(0, t)$, and independent of starting time s.

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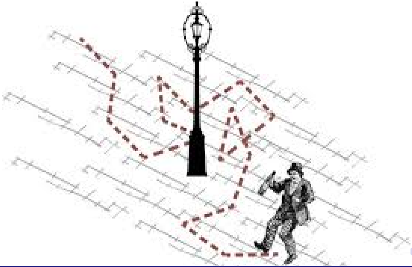
Let's Construct Brownian Motion!

- BM is constructed from random walks

Definition

A **random walk** is a stochastic process formed by successive summation of independent, identically distributed random variables (i.i.d.s).

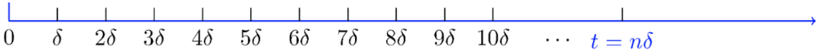
ex: drunkard's walk, a fair die



Theorem

Standard Brownian motion exists, and satisfies the above conditions.

We can divide the real line $[0, \infty)$ into tiny intervals of length δ.
Each sub-interval is a time slot of length δ.
$[0, \delta), [\delta, 2\delta), [2\delta, 3\delta) \ldots [(k-1\delta), k\delta)$ for $k < \infty$.



We toss a fair coin. Random variables X_i :

$$X_i = \begin{cases} +\sqrt{\delta}, & \text{with probability } 1/2, \\ -\sqrt{\delta}, & \text{with probability } 1/2, \end{cases} \tag{0.5}$$

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where X_is are independent (i.i.d's).

$$E(X_i) = 0 \text{ due to being symmetrical} \tag{0.6}$$

$$Var(X_i) = \delta \tag{0.7}$$

$B_0 = 0$

Define $W(t)$ where $W(0) = 0$. Then,

$$W(t) = W(n\delta) = \sum_{i=1}^{n} X_i. \tag{0.8}$$

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Since $W(t)$ is the sum of n i.i.d. variables,

$$E(W(t)) = \sum_{i=1}^{n} E(X_i) \tag{0.9}$$

$$= 0, \tag{0.10}$$

$$Var(W(t)) = \sum_{i=1}^{n} Var(X_i) \tag{0.11}$$

$$= nVar(X_1) \tag{0.12}$$

$$= n\delta \tag{0.13}$$

Independent Increments

For $0 \leq t_1 < t_2 < t_3 \ldots < t_n$,

$$W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots W(t_n) - W(t_{n-1}) \quad (0.15)$$

are independent.

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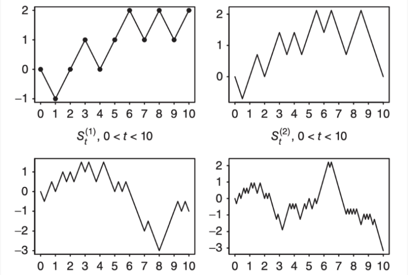
$$W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots W(t_n) - W(t_{n-1}) \quad (0.15)$$

are independent. For $t \in (0, \infty)$, as $n \to \infty$, $\delta \to 0$. By the Central Limit Theorem:

$$W(t) \sim \mathcal{N}(0, t). \quad (0.16)$$



Stationary Increments

$W(t)$ must only depend on the length of the interval:
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For $0 \leq t_1 < t_2$, let $t_1 = n_1\delta$ and $t_2 = n_2\delta$, we see that

$$W(t_1) = \sum_{i=1}^{n_1} X_i, \tag{0.17}$$

$$W(t_2) = \sum_{i=1}^{n_2} X_i. \tag{0.18}$$

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Then, to find the interval in between,

$$W(t_1) - W(t_2) = \sum_{i=1+n_1}^{n_2} X_i. \tag{0.19}$$

$$E(W(t_1) - W(t_2)) = E(\sum_{1+n_1}^{n_2} X_i) \tag{0.20}$$
$$= 0 \tag{0.21}$$

$$Var(W(t_1) - W(t_2)) = Var(\sum_{1+n_1}^{n_2} X_i) \tag{0.22}$$
$$= (n_2 - n_1) Var(X_1) \tag{0.23}$$
$$= (n_2 - n_1)\delta \tag{0.24}$$
$$= t_2 - t_1 \tag{0.25}$$

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Hence, $W(t_2) - W(t_1)$ converges to $\mathcal{N}(0, t_2 - t_1)$, normally distributed with expectation 0 and variance t.

Continuity

$$\mathbb{P}(\omega \in \Omega : B_\omega(t) \text{ is a continuous function of t}) = 1$$

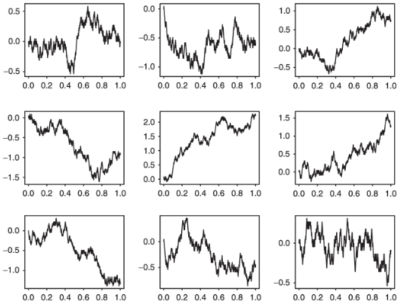

Figure: Sample paths of Brownian motion on $[0, 1]$

Markov and Martingale Properties

Brownian motion is both a Markov process and a Martingale.

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Definition

A Markov process $(X_t)_{t \geq 0}$ can be mathematically represented if

$$P((X_{t+s} \leq y)|X_u, 0 \leq u \leq s) = P((X_{t+s} \leq y)|X_s) \qquad (0.26)$$

for all $s, t > 0$ and real y.

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for all $s, t > 0$ and real y.

This means the probability of state X time t after s only depends on the state at X_s, not anything in between.

Martingale Properties

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2. $E(|Y_t|) < \infty$.

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future expectation of an event is equal to it's current value:

$$E(Y_0) = E(E(Y_t|Y_{0+s}) = E(Y_s).$$

See paper for BM and random walks proof + optional stopping theorem

BM in Quantative Genetics

Mean value of a trait $\overline{z}$, population with size N_e, mutations (random variables) mean 0 and variance σ_m^2. The population evolves purely based on this mutation and genetic drift (random chance). The limit of these random walks, the mean value of the trait, is thus a Brownian motion path as time t increases.

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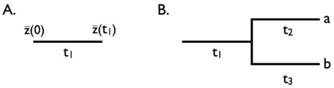
$$\sigma_B^2(t) = \frac{h^2 \sigma_W^2 t}{N_e}$$

Phylogenetic Trees



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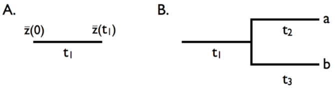

A.

$$\mathbb{E}[\bar{z}_t] \sim N(\bar{z}_0, \sigma_B^2 t_1).$$

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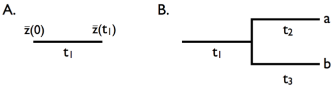

A.

$$\mathbb{E}[\overline{z}_t] \sim N(\overline{z}_0, \sigma_B^2 t_1).$$

B.

$$\overline{x}_a \sim N[\overline{z}_0, \sigma_B^2(t_1 + t_2)],$$

and

$$\overline{x}_b \sim N[\overline{z}_0, \sigma_B^2(t_1 + t_3)].$$

$$\overline{x}_a = \Delta\overline{x}_1 + \Delta\overline{x}_2$$
$$\overline{x}_b = \Delta\overline{x}_1 + \Delta\overline{x}_3$$

$$Cov(\overline{x}_b, \overline{x}_b) = Var(\Delta\overline{x}_1) = \sigma_B^2 t_1. \tag{0.27}$$

$$\begin{bmatrix} \sigma^2(t_1 + t_2) & \sigma^2 t_1 \\ \sigma^2 t_1 & \sigma^2(t_1 + t_3) \end{bmatrix} = \sigma^2 \begin{bmatrix} t_1 + t_2 & t_1 \\ t_1 & t_1 + t_3 \end{bmatrix} = \sigma^2 \mathbf{C}$$

Figure: Variance-Covariance Matrix

More in my paper!

Thank you for listening!