The Gershgorin Circle Theorem

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A matrix is a rectangular array of numbers (or any other mathematical object) . Most commonly, a matrix over a field $\mathbb{F}$. The set of matrices with m rows and n columns over the field $\mathbb{F}$ is denoted as $M_{m\times n}(\mathbb{F})$. For an example, take $A \in M_{3\times 2}(\mathbb{R})$

$$A = \begin{bmatrix} 1 & \pi \\ e & -1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

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Gershgorin Circle theorem

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Then the eigenvalues of A *are in the set*

$$G(A) = \bigcup_{i=1}^{n} G_i(A)$$

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Due to the triangle inequality:

$$|x_p|\,|\lambda - a_{p,p}| \leq \sum_{j \neq p} |a_{p,j}|\,|x_j| \leq |x_p| \sum_{j \neq p} |a_{p,j}| = |x_p|\, R_p(A)$$

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and since $|x_p| > 0$, we reach that $|\lambda - a_{p,p}| \leq R_p(A)$ and therefore $\lambda \in G_p(A)$ and the larger set $G(A)$.

Example

Say we have the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & i & 1 \\ 1 & i & -i-1 \end{bmatrix}$

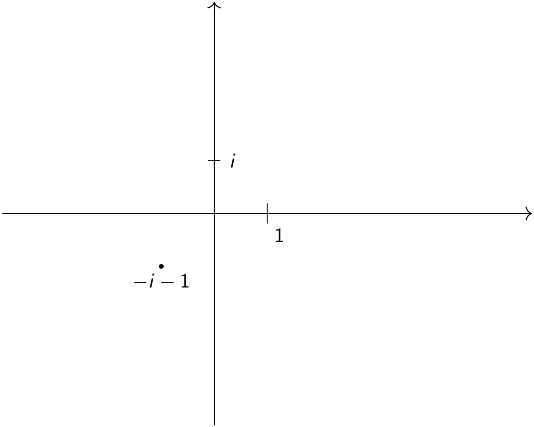

Figure: $G(A)$

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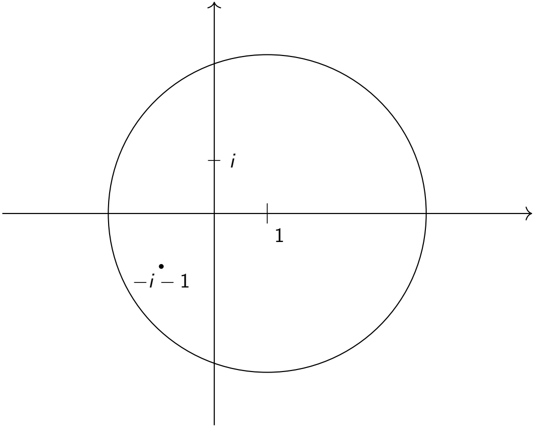

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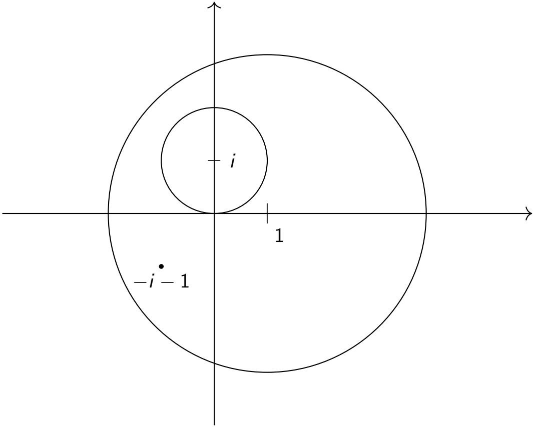

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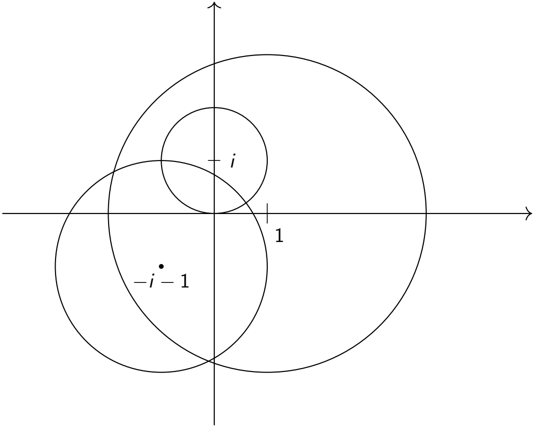

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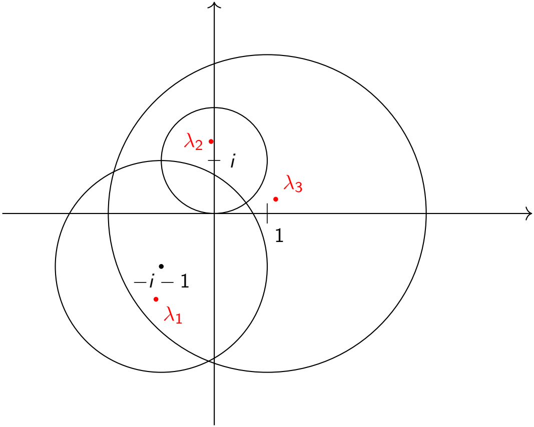

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If k Gershgorin circles intersect, then there are k eigenvalues in that area.

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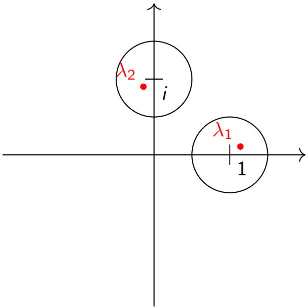

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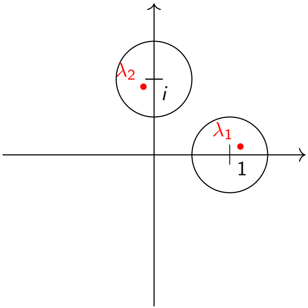

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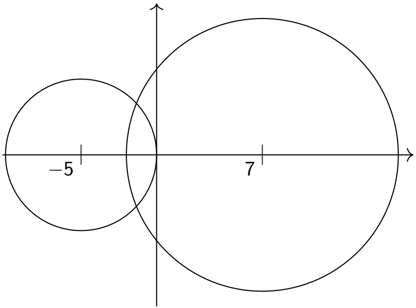

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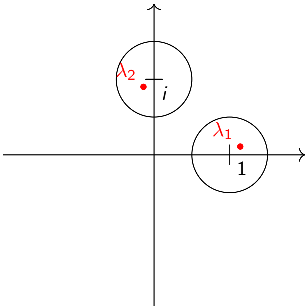

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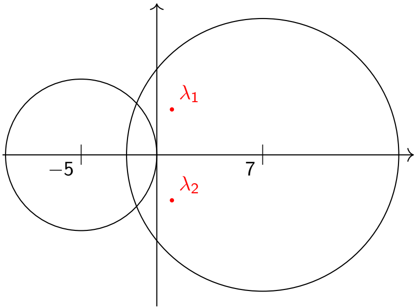

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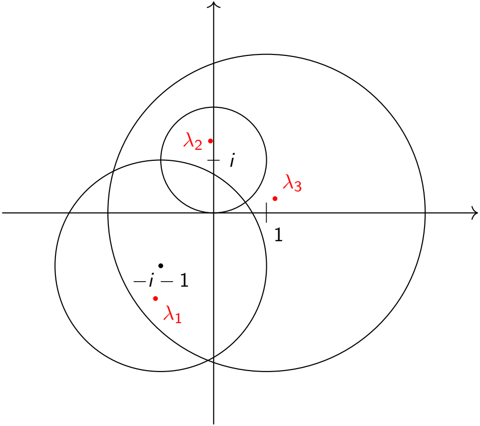
Turns out, there is, $S^{-1}AS$ has the same eigenvalues as A. We can take

$$S = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{bmatrix} \text{ with } p_1, p_2, \dots, p_n \in \mathbb{R}_{>0}.$$

Example

We take $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & i & 1 \\ 1 & i & -i-1 \end{bmatrix}$ and $S = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, we get



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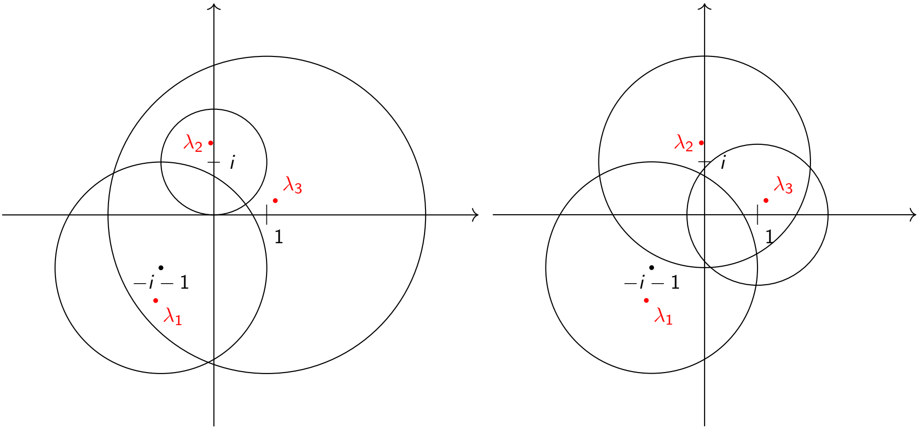

(a) $G(A)$ (b) $G(S^{-1}AS)$

A perturbed matrix

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This is the main question of my paper. If we look at matrices with certain properties, we get quite some interesting results.

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Well, every eigenvalue of $D + E$ is in the set

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therefore, if $\hat{\lambda}$ is an eigenvalue of $D + E$ there is an eigenvalue of D such that

$$\left| \hat{\lambda} - \lambda \right| \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |e_{i,j}|.$$

Thank you!

Thank you for listening to my talk!