ON THE PERTURBATION OF EIGENVALUES

DRAGOȘ GLIGOR

1. Introduction

The study of eigenvalues has long been a subject of great interest for both applied and pure mathematicians. In 1931, Soviet mathematician Semyon Aranovich Gershgorin published a paper that proved a sensational result: the eigenvalues of a matrix lie in a collection of discs in the complex plane, which captured the attention of many mathematicians and was quickly expanded upon. This is the main theorem we will discuss, as well as the perturbation bounds facilitated by it.

We begin with sections on important preliminary knowledge that will be helpful, if not outright necessary for understanding and talking about our subject. If you are already comfortable with the concepts discussed in sections 2, 3, and 4, feel free to skip them.

Section 5 discusses the Gershgorin circle theorem, introducing ways to help improve the approximation and notions that are related to it, as well as featuring plenty of visualizations to showcase its inherent geometric beauty.

We conclude in section 6 by discussing what happens to the eigenvalues of a matrix if we perturb it, as in, adding another relatively small matrix to it. How much do the eigenvalues change, and can we approximate the eigenvalues of either the original matrix or the perturbed one by knowing the eigenvalues of the other? These are all questions we shall answer.

2. Permutations

Definition 2.1. A permutation of degree n is a bijective function $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

The set of all permutations of degree n is noted as S_n. The cardinal of S_n is $n!$. A permutation of degree n is often represented as such:

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & k & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(k) & \ldots & \sigma(n) \end{pmatrix}$$

Permutations are usually denoted using lowercase Greek letters. A permutation of particular interest is the identity permutation, denoted by ϵ, defined as:

$$\epsilon = \begin{pmatrix} 1 & 2 & \ldots & k & \ldots & n \\ 1 & 2 & \ldots & k & \ldots & n \end{pmatrix}$$

In short, $\epsilon(k) = k$ for all $k \in \{1, 2, \ldots, n\}$.

Definition 2.2. The product of two permutations $\sigma, \tau \in S_n$ is the composition of their respective function. More clearly $\sigma\tau = \sigma \circ \tau = \sigma(\tau(x))$ for all $x \in \{1, 2, \ldots, n\}$. The multiplication of two permutations has the following properties for any $\alpha, \beta, \gamma \in S_n$:

(1) *Associativity*: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

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(2) *Neutral element*: There is a permutation such that $\alpha\epsilon = \epsilon\alpha = \alpha$ for all $\alpha \in S_n$. The identity element for the multiplication of permutations is the identity permutation ϵ.
(3) *Inverse*: for any permutation α, there exist a permutation α^{-1} such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \epsilon$ (the identity element). α^{-1} is called the inverse of permutation α.

Remark 2.1. A quick way of finding the inverse of a permutation $\sigma \in S_n$ represented as such $\sigma = \begin{pmatrix} 1 & 2 & \dots & k & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(k) & \dots & \sigma(n) \end{pmatrix}$ is by switching the top and bottom rows $\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(k) & \dots & \sigma(n) \\ 1 & 2 & \dots & k & \dots & n \end{pmatrix}$ and then ordering the top row increasingly.

Definition 2.3. For a permutation $\sigma \in S_n$, an inversion is a pair (i, j) such that $\sigma(i) > \sigma(j)$. The number of inversions of the permutation σ is noted as $m(\sigma)$.

As a short note, the smallest deviation(as in, the one that changes the least amount of elements) from ϵ is an inversion. This is intuitive, for if you try to change only one element, the function would no longer be a bijection.

Definition 2.4. The signature of a permutation $\sigma \in S_n$ is defined as -1 to the power of the number of inversions and noted as $sgn(\sigma) = (-1)^{m(\sigma)}$
If $sgn(\sigma) = 1$ then it is called an even permutation.
If $sgn(\sigma) = -1$ then it is called an odd permutation.

Example. For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ we have the inversions $(1, 3), (2, 3), (2, 4)$. As such $m(\sigma) = 3$ and $sgn(\sigma) = -1$ and thus σ is an odd permutation.

3. The Characteristic polynomial of a matrix

If you are reading this paper, then you are already familiar with the equation that defines the eigenvalues of a matrix:

$$Ax = \lambda x$$

where (λ, x) is an eigenpair of A and $A \in M_n(\mathbb{C})$, the set of matrices of size $n \times n$ with elements in the complex plane. Also recalled that $\sigma(A)$ denotes the set of the eigenvalues of A.
It does not take much to see that this equation is equivalent to $\lambda x - Ax = 0_{n\times 1} \Leftrightarrow (\lambda I_n - A)x = 0_{n\times 1}$. Since the vector $0_{n\times 1}$ is by definition not considered an eigenvector, then we conclude that the matrix $\lambda I_n - A$ is singular. As such, $\det(\lambda I_n - A) = 0$.

Definition 3.1. For a matrix $A \in M_n(\mathbb{C})$ the determinant

$$p_A(t) = \det(tI_n - A), t \in \mathbb{C},$$

is the characteristic polynomial of the matrix A. We refer to the equation $p_A(t) = 0$ as the characteristic equation of the matrix A.

Observation 3.1. The characteristic polynomial of the matrix $A = [a_{i,j}]$ is a polynomial of degree n and $p_A(t) = t^n - \mathrm{Tr}(A)t^{n-1} + \dots + (-1)^n \det(A)$. Moreover, $p_A(t) = 0$ if and only if $\lambda \in \sigma(A)$.

Proof. The definition of a determinant is $\det(A) = \sum_{\tau \in S_n} sgn(\tau) \prod_{i=1}^{n} a_{i,\tau(i)}$. As such, each element of the sum is the product of n elements of $tI_n - A$, which has the following form

$$tI_n - A = \begin{bmatrix} t - a_{1,1} & -a_{1,2} & \dots & -a_{1,n} \\ -a_{2,1} & t - a_{2,2} & \dots & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \dots & t - a_{n,n} \end{bmatrix}$$

To be more explicit, the diagonal elements are of the form $t - a_{i,i}$ and the off-diagonal elements are of the form $-a_{i,j}$. The product of n elements of this matrix produces a polynomial of t with a maximum degree of n. Moreover, the only summand of degree n is the product of the elements of the matrix's diagonal, which corresponds to the identity permutation ϵ:

$$(t - a_{1,1})(t - a_{2,2}) \dots (t - a_{n,n}) = t^n - (a_{1,1} + a_{2,2} + \dots + a_{n,n})t^{n-1} + \dots$$

Any other summand will include at least one $-a_{i,j}$ term, corresponding to a permutation different from ϵ. As such, the summand cannot include the terms $t - a_{i,i}$ and $t - a_{j,j}$, and will therefore have a maximum degree of $n - 2$.
For $t = 0$, we have $\det(-A) = (-1)^n \det(A)$, which is the constant term.
With $\mathrm{Tr}(A) = a_{1,1} + a_{2,2} + a_{3,3} + \dots a_{n,n}$, we reach the form

$$p_A(t) = \det(tI_n - A) = t^n - \mathrm{Tr}(A)t^{n-1} + \dots + (-1)^n \det(A)$$

and the first conclusion of our observation has been proved.
As previously stated, $\det(\lambda I_n - A) = 0$ for all eigenvalues of A. This can also be written as $p_A(\lambda) = 0$. Therefore, λ is a root of the polynomial $p_A(t)$. It follows that all eigenvalues of A are roots of $p_A(t)$, and since $p_A(t)$ is a polynomial of degree n, which means it has exactly n roots in the complex plane and A has n eigenvalues, then all the roots of $p_A(t)$ are eigenvalues of A. ■

Definition 3.2. The algebraic multiplicity of an eigenvalue of A is its multiplicity as a root of $p_A(t)$.

Due to the property of the roots of the characteristic polynomial coinciding with the eigenvalues of A, it is an important notion in the study of eigenvalues.

Example. Say you want to find the eigenvalues of the matrix $A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}$. For a matrix of size 2, its characteristic polynomial has the rather simple form: $p_A(t) = t^2 - \mathrm{Tr}(A)t + \det(A)$. For our example, $p_A(t) = t^2 - 10t + 25 = (t-5)^2$. As it is clear, the roots of this polynomial are 5 and well, 5. As such, $\sigma(A) = \{5\}$, and the eigenvalue 5 has an algebraic multiplicity of 2.

4. Vector and matrix norms

Definition 4.1. Let V be a vector space over the field $\mathbb{C}$. A function $\|\cdot\| : V \to \mathbb{R}$ is a norm (sometimes called a vector norm) if, for all $x, y \in V$ and all $c \in \mathbb{C}$,

(1) $\|x\| \geq 0$
(1a) $\|x\| = 0$ *if and only if* $x = 0$
(2) $\|cx\| = |c| \, \|x\|$
(3) $\|x + y\| \leq \|x\| + \|y\|$

Examples of vector norms for a vector $x = [x_1 \dots, x_n]^T \in \mathbb{C}^n$ that may be used in this paper are:

$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$ (the l_p-norm)
$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$ (the max norm or l_∞-norm)

Definition 4.2. A function $|||\cdot||| : M_n \to \mathbb{R}$ is a matrix norm if, for all A, $B \in M_n$, it satisfies the same 4 properties as a vector norm in addition to the following one:

$$|||AB||| \leq |||A|||\,|||B||| \tag{4}$$

The matrix norms that may be used for any matrix in this paper $A \in M_n$ are:

$|||A|||_1 = \max_{1\leq 2\leq n} \sum_{i=1}^{n} |a_{i,j}|$ (the maximum column sum norm)
$|||A|||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{i,j}|$(the maximum row sum norm)
$|||A|||_2 = \max_{\|x\|_2=1} \|Ax\|_2$ where x is a vector (the spectral matrix norm). It is also equivalent to the largest singular value of the matrix A.
$|||A|||_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{i,j}|^2} = \sqrt{\mathrm{Tr}(A^*A)}$ where $A^* = \overline{A^T}$ denotes the conjugate transpose of the matrix A(the Frobenius norm)

Observation 4.1. The Frobenius norm of a matrix $A \in M_n$ is the same as its conjugate transpose:

$$|||A|||_F = |||A^*|||_F$$

Proof. $|||A^*|||_F = \sqrt{\mathrm{Tr}((A^*)^*A^*} = \sqrt{\mathrm{Tr}(AA^*)} = \sqrt{\mathrm{Tr}(A^*A)} = |||A|||_F$ ■

The Frobenius norm has some interesting properties, that will be useful in the last chapter of this paper. To express them, we must first define another notion.

Definition 4.3. A matrix $A \in M_n$ is unitary if it satisfies the condition $A^*A = AA^* = I_n$, as in its conjugate transpose is its own inverse.

Remark 4.2. As it is observable in the definition, the conjugate transpose of a unitary matrix is also unitary.

Lemma 4.3. *The Frobenius norm $|||\cdot|||_F$ is unitary invariable. As in, if $A \in M_n$ and a unitary matrix $U \in M_n$, then $|||AU|||_F = |||UA|||_F = |||A|||_F$.*

Proof. Let $A \in M_n$ and $U \in M_n$. Therefore

$$|||UA|||_F = \sqrt{\mathrm{Tr}((UA)^*(UA))} = \sqrt{\mathrm{Tr}(A^*U^*UA) = Tr(A^*A)} = |||A|||_F$$

and since U^* is also unitary

$$|||AU|||_F = |||(U^*A^*)^*|||_F = |||U^*A^*|||_F = |||A^*|||_F = |||A|||_F$$

■

Definition 4.4. The condition number of matrix norm $|||\cdot|||$ for a matrix $S \in M_n$, denoted as $\kappa(S)$, is defined as

$$\kappa(S) = \begin{cases} |||S^{-1}|||\,|||S||| & if\ S\ is\ nonsingular \\ \infty & if\ S\ is\ singular \end{cases}$$

Observation 4.4. If $A \in M_n$ and $|||\cdot|||$ such that $|||A||| < 1$, then $I_n - A$ is nonsingular.

Proof. Let $A \in M_n$ such that $|||A||| < 1$ let us assume that $I_n - A$ is singular. Then there exists a vector x such that $(I_n - A)x = 0$. Then $x - Ax = 0 \Leftrightarrow x = Ax$. Applying the norm we get $|||x||| = |||Ax||| \leq |||A|||\,|||x|||$ and since x is a nonzero vector we get that $|||x||| > 0$ and therefore $1 \leq |||A|||$, which is a contradiction. ■

5. The Gershgorin Circle theorem

Theorem 5.1 (Gershgoin Circle Theorem). *[Ger31] Let $A = [a_{i,j}] \in M_n$ and let*

$$R_i(A) = \sum_{j \neq i} |a_{i,j}|\,,\ \ i \in \{1, 2, \ldots, n\}$$

denote the deleted absolute row sums of A, and consider the following n Gershgorin discs defined as:

$$\{z \in \mathbb{C} \mid\ |z - a_{i,i}| \leq R_i(A)\},\ i \in \{1, 2, \ldots, n\}$$

The eigenvalues of A are in the union of Gershgorin discs

$$G(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} \mid\ |z - a_{i,i}| \leq R_i(A)\}.$$

Proof. Consider the eigenpair (λ, x) such that $Ax = \lambda x$ and $x = [x_i] \neq 0$. Let $p \in \{1, 2, \ldots, n\}$ such that $|x_p| = \|x\|_\infty$. Therefore, $|x_p| \geq |x_i|$, for all $i \in \{1, 2, \ldots, n\}$. Equating the p-th entries of the equation $Ax = \lambda x$ gives us $\lambda x_p = \sum_{j=1}^{n} a_{p,j} x_j$, which will be written as:

$$x_p(\lambda - a_{p,p}) = \sum_{j \neq p} a_{p,j} x_j.$$

Due to the triangle inequality and how we chose p:

$$|x_p|\,|\lambda - a_{p,p}| = \left| \sum_{j \neq p} a_{p,j} x_j \right| \leq \sum_{j \neq p} |a_{p,j} x_j| = \sum_{j \neq p} |a_{p,j}|\,|x_j| \leq |x_p| \sum_{j \neq p} |a_{p,j}| = |x_p|\, R_p(A)$$

Since $|x_p| \neq 0$, we reach that $|\lambda - a_{p,p}| \leq R_p(A)$ and therefore $\lambda \in \{z \in \mathbb{C} \mid\ |z - a_{p,p}| \leq R_p(A)\}$ and the larger set $G(A)$. ■

This theorem brings a geometric element to the location of eigenvalues. Each set describes a disc in the complex plane with center $a_{i,i}$ and radius $R_i(A)$. As such, $G(A)$ can very easily be visualized.

Corollary 5.2. *Let $C_j(A) = \sum_{j \neq i} |a_{i,j}|$ denote the absolute deleted column sum of a column j of the matrix A. Then the eigenvalues of A are in the union of Gershgorin discs*

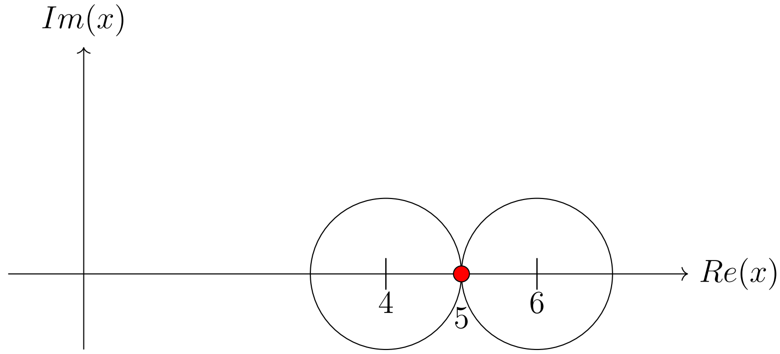
$$\bigcup_{j=1}^{n} \{z \in \mathbb{C} \mid\ |z - a_{j,j}| \leq C_j(A)\}\ .$$

Proof. Apply Theorem 5.1 to the matrix A^T, where A^T denotes the transpose of A. As A^T has the same eigenvalues as A, we reach the conclusion outlined above. ■

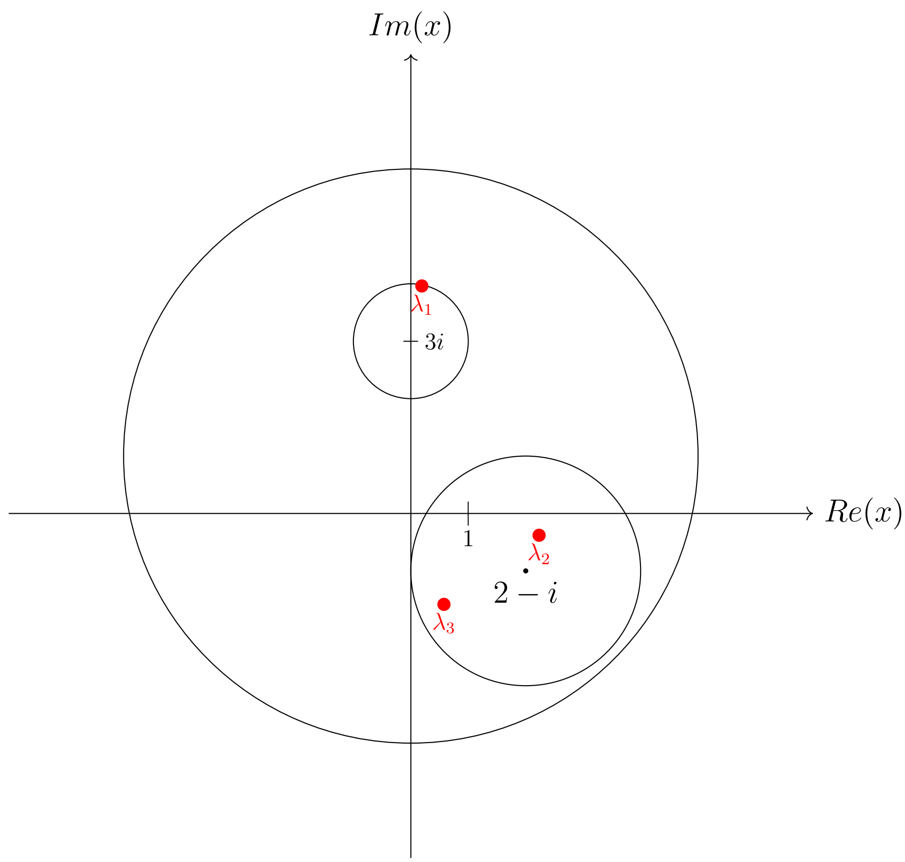
Corollary 5.3. *[HJ13] If the union of k discs form a set $G_k(A)$ that is disjoint from the remaining $n - k$ discs, then $G_k(A)$ contains exactly k eigenvalues, counted with their algebraic multiplicities.*

The proof of this corollary is outside the scope of this paper. Refer to Chapter 6.1, page 388, Theorem 6.1.1 in the cited work for a proof. As the theorems and corollaries discussed so far have an intrinsic geometric element, they are best understood with the help of visuals.

Example. Using the matrix as in the example in section 3, $A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}$. Using Theorem 5.1. we get the following picture. We denote the positions of the eigenvalues with red.

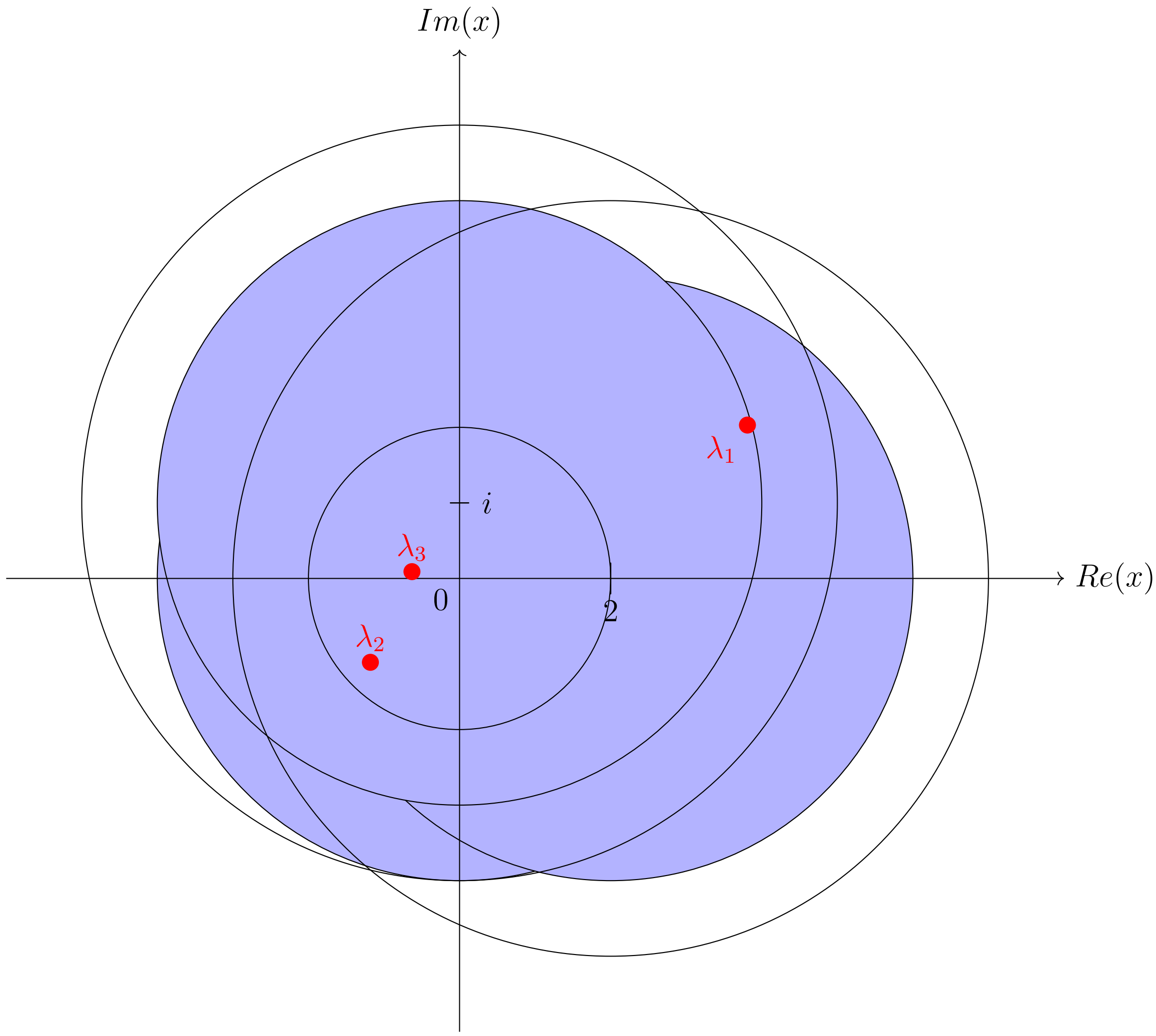


For a more complicated example, we could take a look at the matrix $B = \begin{bmatrix} 3i & 0 & -1 \\ i & 2-i & 1 \\ 4 & 0 & 1 \end{bmatrix}$:



The eigenvalues of B are a lot "messier" than those of A, and as such do not lie neatly on the boundaries of the Gershgorin circles. While λ_1 may appear so visually, upon closer inspection we find that $\lambda_1 \approx 0.191543 + 3.96242i$ which is not on the circle's bound.

To exemplify corollary 5.2, let's take the matrix $C = \begin{bmatrix} i & 3 & 2 \\ 3i & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix}$. It gives us a nice image, where the area shaded with blue represents $G(C^T)$ while the non-shaded circles represent $G(A)$.



While Theorem 5.1 approximates nicely where the eigenvalues lie, we would still like to try and do better. One way could be to take the set $G(A)$ and $G(A^T)$ to see which gives us a better approximation, as we did above, since again, A^T and A have the same eigenvalues. Another pathway is shown to us by the following lemma.

Lemma 5.4. *Let A,$S \in M_n$ and S nonsingular. Then, $S^{-1}AS$ has the same eigenvalues as A, where S^{-1} denotes the inverse of the matrix S.*

Proof. We shall prove this by proving they have the same characteristic polynomial and using the properties of determinants.

$$\begin{aligned}p_A(t) &= \det(tI_n - A) = \det(I_n)\det(tI_n - A) = \det(S^{-1}S)\det(tI_n - A)\\ &= \det(S^{-1})\det(tI_n - A)\det(S) = \det(S^{-1}(tI_n - A)S)\\ &= \det(tS^{-1}S - S^{-1}AS) = \det(tI_n - S^{-1}AS) = p_{S^{-1}AS}(t)\end{aligned}$$

As the roots of the characteristic polynomial are equivalent to the eigenvalues, and since $p_A(t) = p_{S^{-1}AS}(t)$ we reach that A and $S^{-1}AS$ have the same eigenvalues. ■

Using this lemma, we can choose a S such $S^{-1}AS$ gives us more favorable Gershgorin circles. In particular, choosing $S = D = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{bmatrix}$ with all $p_i > 0$. The inverse is of the form $D^{-1} = \begin{bmatrix} \frac{1}{p_1} & 0 & \dots & 0 \\ 0 & \frac{1}{p_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{p_n} \end{bmatrix}$ and we reach that $D^{-1}AD = \begin{bmatrix} a_{1,1} & \frac{a_{1,2}p_2}{p_1} & \dots & \frac{a_{1,n}p_n}{p_1} \\ \frac{a_{2,1}p_1}{p_2} & a_{2,2} & \dots & \frac{a_{2,n}p_n}{p_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n,1}p_1}{p_n} & \frac{a_{n,2}p_2}{p} & \dots & a_{n,n} \end{bmatrix}$.

To be explicit, any element of this matrix is of the form $\frac{a_{i,j}p_j}{p_i}$ where i represents the row index and j the column index of the matrix. Applying Gershgorin's circle theorem to it, we reach the following result.

Corollary 5.5. *Let $A = [a_{i,j}] \in M_n$ and let $p1, p2, ..., p_n \in \mathbb{R}_{>0}$. The eigenvalues of A are in the union of n discs*

$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} \mid |z - a_{i,i}| \leq \frac{1}{p_i} \sum_{j \neq i} p_j \, |a_{i,j}| \right\} = G(D^{-1}AD).$$

Remark 5.6. Corollary 5.2. and 5.3. can also be applied to this matrix. In particular, applying 5.2. implies the eigenvalues of A lie in the set

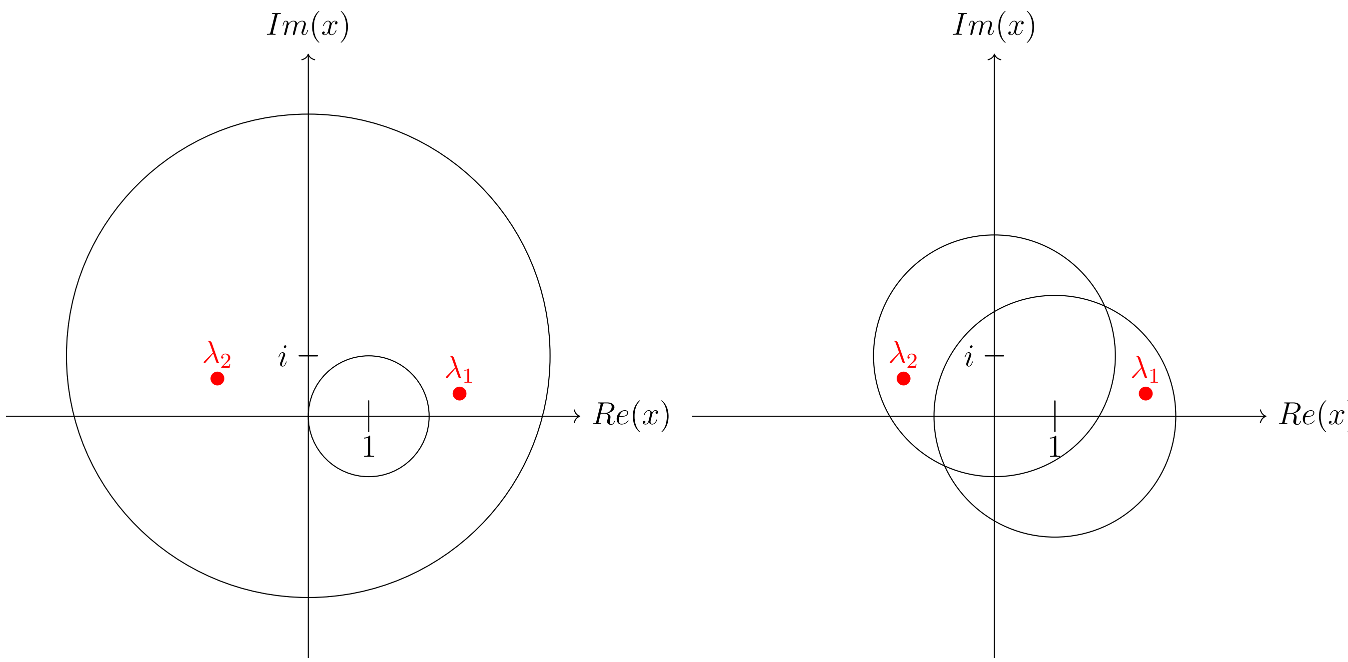
$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} \mid |z - a_{i,i}| \leq p_j \sum_{j \neq i} \frac{1}{p_i} \, |a_{i,j}| \right\} = G((D^{-1}AD)^T) = G(DA^TD^{-1}).$$

Example. Say we have the matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & i \end{bmatrix}$. If we were to apply Theorem 5.1. directly, we would get a rather wide approximation. If we instead apply it using corollary 5.5. taking $p_1 = 1$ and $p_2 = 2$ we get a more accurate approximation.

Another method for getting tighter approximations for certain matrixes is revealed in the paper [BS17]. Before we reach the main result, we need to define another notion.

Definition 5.1. Let $A = [a_{i,j}] \in M_n$ be a matrix and λ an eigenvalue of A. The dimension of the eigenspace of A associated with λ is the geometric multiplicity of λ.

In simpler terms, the geometric multiplicity of an eigenvalue λ is the number of linearly independent vectors x such that $Ax = \lambda x$.



(a) Using Theorem 5.1. **(b)** Using Corollary 5.5.

Figure 1. The corresponding Gershgorin circles for the matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & i \end{bmatrix}$

Theorem 5.7. *[BS17] Let $A = [a_{i,j}] \in M_n(\mathbb{R}_{\geq 0})$ and λ an eigenvalue of A with geometric multiplicity of at least 2 and let $r_i(A)$ denote the absolute deleted row sum of the largest $\lfloor \frac{n}{/2} \rfloor$ terms. Then λ is in a half Gershgorin disc $\{z \in \mathbb{C} \mid |z - a_{i,i}| \leq r_i(A)\}$, for some i.*

The proof for this theorem is outside the scope of this paper and can be found in the cited work.

Example. Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. This matrix has the $p_A(t) = (t-5)(t-2)^2$ so it has the eigenvalues $\lambda_1 = 5$ and $\lambda_{2,3} = 2$ with an algebraic multiplicity of 2. The linearly independent vectors that verify $Ax = 2x$ are $v_1 = [-1, 0, 1]^T$ and $v_2 = [-1, 1, 0]^T$ and such 2 has an geometric multiplicity of 2 and we can apply 5.7.

Definition 5.2. A matrix $A = [a_{i,j}] \in M_n$ is diagonally dominant if

$$|a_{i,i}| \geq \sum_{j \neq i} |a_{i,j}| = R_i(A) \textit{ for all } i \in \{1, 2, \ldots, n\}$$

and strictly diagonally dominant if

$$|a_{i,i}| > \sum_{j \neq i} |a_{i,j}| = R_i(A) \textit{ for all } i \in \{1, 2, \ldots, n\}.$$

Theorem 5.8. *Let a matrix $A = [a_{i,j}] \in M_n$ be strictly diagonally dominant. Then A is nonsingular.*

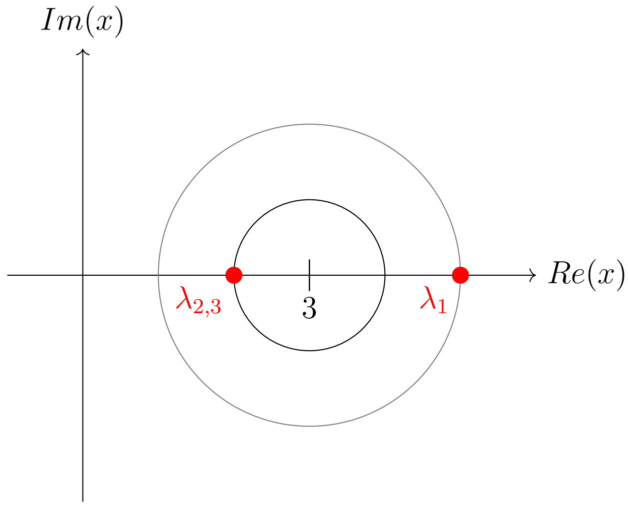


Figure 2. The half-circle of the matrix A

Proof. Let us take a matrix $A = [a_{i,j}] \in M_n$ that is strictly diagonally dominant and assume it is singular. That implies $\lambda = 0$ is one of its eigenvalues. Using the Gershgorin circle theorem, we get that for an eigenvalue λ that is in the i-th circle

$$|\lambda - a_{i,i}| \leq R_i(A) \tag{1}$$

but strict diagonal dominance implies

$$R_i(A) < |a_{i,i}| \tag{2}$$

Using (1) and (2) we get

$$|\lambda - a_{i,i}| < |a_{i,i}| \tag{3}$$

As we made no assumption on the nature of λ and i, this applies to any eigenvalue in the Gershgorin circle where they are located. As such, the same would apply to $\lambda = 0$ that is located in the k-th Gershgorin circle. Applying (3) we reach that $|a_{k,k}| < |a_{k,k}|$, which is a contradiction. Therefore, the matrix A is nonsingular.

■

One may wonder if non-strict diagonal dominance is enough to imply nonsingularity. Unfortunately, the quick counterexample $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ dispels that notion.

Lemma 5.9. *Let $A = [a_{i,j}]$, and let $z \in \mathbb{C}$ be given, then*

(a) *z is not in the interior of any Gershgorin disc if and only if*

$$|z - a_{i,i}| \geq R_i(A), \text{ for all } i \in \{1, 2, \ldots, n\}. \tag{5.9a}$$

(b) *z is on the boundary of $G(A)$, then it satisfies the inequality* 5.9*a*.

(c) *A is diagonally dominant, if and only if $z = 0$ satisfies* 5.9*a*.

Proof. (a) z is in the interior of a Gershgorin circle if and only if there exists an $i \in \{1, 2, \ldots, n\}$ such that $|z - a_{i,i}| < R_i(A)$. Its direct negation is the conclusion we want to reach, and as the initial statement is true, so is its direct negation.

(b) z is on the boundary, then it must satisfy $|z - a_{i,i}| = R_i(A)$, for a $k \in \{1, 2, \ldots, n\}$ number of $i's \in \{1, 2, \ldots, n\}$ and $|z - a_{i,i}| > R_i(A)$, *for* $n - k$ *number of* $i's \in \{1, 2, \ldots, n\}$. As such, it satisfies 5.9a.

(c) Let A be diagonally dominant, and let's assume $z = 0$ does not satisfy 5.9a. That would imply that there is an i such that $|a_{i,i}| < R_i(A)$, which is a contradiction. As such, if A is diagonally dominant, $z = 0$ must satisfy 5.9a.

■

Lemma 5.10. *Let* (λ, x) *be an eigenpair of* $A = [a_{i,j}]$ *and suppose that* λ *satisfies* 5.9a. *Then*

(a) *if* $p \in \{1, 2, \ldots, n\}$ *satisfies* $|x_p| = \|x\|_\infty$ *then* $|\lambda - a_{p,p}| = R_p(A)$. *To be clear, this means that the* p*-th Gershgorin circle of* A *passes through* λ.

(b) *if* $p, q \in \{1, 2, \ldots, n\}$, $|x_p| = \|x\|_\infty$ *and* $a_{p,q} \neq 0$ *then* $|x_q| = \|x\|_\infty$.

Proof. Take a $p \in \{1, 2 \ldots, n\}$ such that $|x_p| = \|x\|_\infty$. Then using Theorem 5.1

$$|\lambda - a_{p,p}| \, \|x\|_\infty = |\lambda - a_{p,p}| \, |x_p| = \left| \sum_{j \neq p} a_{p,j} x_j \right| \leq \sum_{j \neq p} |a_{p,j}| \, |x_j| \leq \sum_{j \neq p} |a_{p,j}| \, \|x\|_\infty = R_p(A) \, \|x\|_\infty .$$

Therefore $|\lambda - a_{p,p}| \leq R_p(A)$, but λ satisfies 5.9a, and we reach that $|\lambda - a_{p,p}| = R_p(A)$ and as such, conclusion (a). Revisiting the relation above, we get that

$$|\lambda - a_{p,p}| \, \|x\|_\infty = \sum_{j \neq p} |a_{p,j}| \, |x_j| = \sum_{j \neq p} |a_{p,j}| \, \|x\|_\infty = R_p(A) \, \|x\|_\infty .$$

Particularly, the equality $\sum_{j \neq p} |a_{p,j}| \, |x_j| = \sum_{j \neq p} |a_{p,j}| \, \|x\|_\infty$ leads us to

$$\sum_{j \neq p} |a_{p,j}| \left(\|x\|_\infty - |x_j| \right) = 0.$$

As each element of the sum is nonnegative, they must all be 0. $a_{p,q} \neq 0$ leads to $|x_q| = \|x\|_\infty$ and conclusion (b). ■

Theorem 5.11. *Let* $A \in M_n$ *and* (λ, x) *an eigenpair of* A *such that* λ *satisfies* 5.9a. *If every entry of* A *is nonzero, then*

(a) *Every Gershgorin circle of* A *passes through* λ.

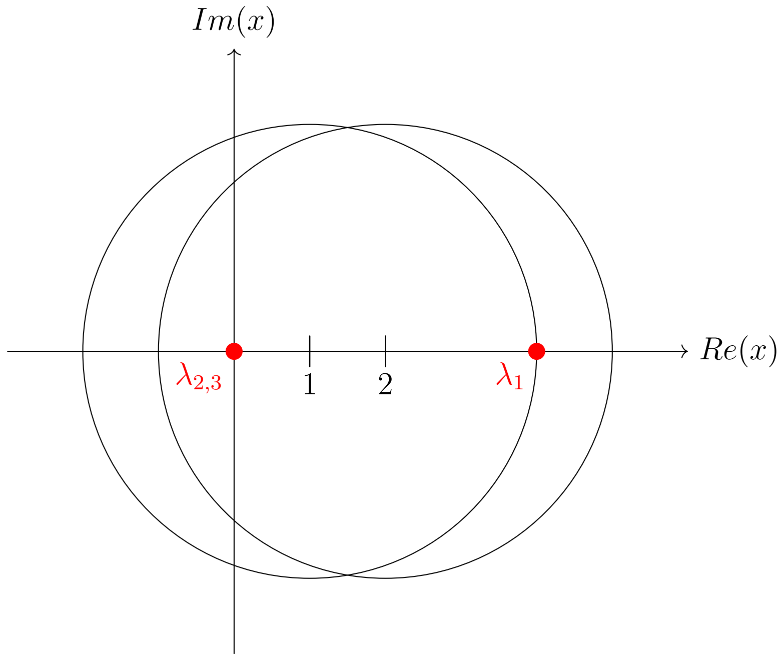
(b) $|x_i| = \|x\|_\infty$ *for all* $i \in \{1, 2, \ldots, n\}$.

Proof. We take $p \in \{1, 2, \ldots, n\}$ such that $|x_p| = \|x\|_\infty$. Applying lemma 5.10 (b) for each $i \in \{1, 2 \ldots, n\} \setminus \{p\}$ we reach conclusion (b). Since $|x_i| = \|x\|_\infty$ for all $i \in \{1, 2 \ldots, n\}$, we can apply lemma 5.10(a) for each i and reach conclusion (a). ■

Example. Say we have the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 4$, $\lambda_{2,3} = 0$. As such, λ_1 satisfies 5.9a. Taking a look at their Gershgorin disks we get the following image:

As you can see, both Gershgorin circles pass through λ_1.

Corollary 5.12. *Let* $A = [a_{ij}] \in M_n$ *with every entry of* A *nonzero. If* A *is diagonally dominant and there is a* $k \in \{1, 2, \ldots, n\}$ *such that* $|a_{k,k}| > R_k(A)$ *then* A *is nonsingular.*



Proof. Since A is diagonally dominant, that means that $z = 0$ satisfies the inequalities $5.9a$. Assume 0 was an eigenvalue of A. Our hypothesis ensures that the previous theorem would apply to it and that every Gershgorin circle would pass through 0. However, $|a_{k,k}| > R_k(A)$ tells us that the k-th Gershgorin circle does not pass through 0. Therefore, 0 is not an eigenvalue of and as such the matrix A is nonsingular. ∎

Next, we will introduce a notion that might seem strange at first glance.

Definition 5.3. A matrix $A \in M_n$ is said to have property SC if for each pair of distinct integers $p, q \in \{1, 2, \ldots, n\}$ there is a sequence of distinct integers $k_1 = p, k_2, \ldots, k_m = q$ such that each entry of $a_{k_1,k_2}, a_{k_2,k_3} \ldots, a_{k_{m-1},k_m}$ is nonzero, where m is the number of integers in the sequence.

Example. Say we have the matrices $A = \begin{bmatrix} 0 & 4 & 5 \\ 3 & 0 & i \\ 2 & -i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} i & -2 & 4 \\ 0 & 4 & 6 \\ 3i+2 & 0 & 7 \end{bmatrix}$. Since the definition asks for a sequence of distinct integers, the entries of $a_{k_1,k_2},\ a_{k_2,k_3}$ will not include diagonal elements of the matrix(this goes for any matrix) and as all the off-diagonal elements are nonzero, it is clear that A has property SC. On the other hand for $p = 1$ and $q = 2$, the only possible sequence is $k_1 = 1,\ k_2 = 3,\ k_3 = 2$ and the entries $b_{k_1,k_2},\ b_{k_2,k_3}$ are equal to $b_{1,3}, b_{3,2}$, which are not all nonzero since $b_{3,2} = 0$ and thus B does not have property SC.

Using the notion of property SC, we can improve the previous theorem and corollary.

Theorem 5.13. *Let $A = [a_{i,j}] \in M_n$ and (λ, x) an eigenpair such that λ satisfies $5.9a$. If A has property SC, then:*

(a) *every Gershgorin disc passes through λ.*
(b) *$|x_i| = \|x\|_\infty$ for all $i \in \{1, 2, \ldots, n\}$.*

Proof. Let $p \in \{1, 2, \dots, n\}$ such that $|x_p| = \|x\|_\infty$. Applying 5.10(a) gives us that $|\lambda - a_{p,p}| = R_p(A)$ so the p-th Gershgorin circle passes through A. Let $q \in \{1, 2, \dots, n\} \setminus \{p\}$. Because A has SC property, then there is a sequence of distinct integers $k_1 = p, k_2, \dots, k_m = q$ such that each entry $a_{k_1,k_2}, a_{k_2,k_3}, \dots, a_{k_{m-1},k_m}$ is nonzero. Since $a_{k_1,k_2} \neq 0$, applying 5.10(b) tells us that $|x_{k_2}| = \|x\|_\infty$ and applying 5.10(a) again gives us $|\lambda - a_{k_2,k_2}| = R_k(A)$. Repeating this process for a_{k_2,k_3} and so forth gives us both conclusion (a) and conclusion (b). ■

Corollary 5.14. *Let $A = [a_{ij}] \in M_n$ and A has property SC. If A is diagonally dominant and there is a $k \in \{1, 2, \dots, n\}$ such that $|a_{k,k}| > R_k(A)$ then A is nonsingular.*

Proof. The proof is the same as that for 5.12, but referring to theorem 5.13 instead of theorem 5.11. ■

Now we will have a series of definitions to help us visualize and more concretely deal with property SC. If you are familiar with directed graphs, then most of the definitions will already be known.

Definition 5.4. For any given matrix $A = [a_{i,j}] \in M_{m,n}$ define $|A| = [|a_{i,j}|]$ and $M(A) = [u_{i,j}]$, in which $u_{i,j} = 1$ if $a_{i,j} \neq 0$ and $u_{i,j} = 0$ if $a_{i,j} = 0$. The matrix $M(A)$ is the indicator matrix of A.

Definition 5.5. The directed graph of $A \in M_n$, denoted by $\Gamma(A)$ is the directed graph of n nodes $P_1, P_2, \dots, P_n$ such that there is a directed arc in $\Gamma(A)$ from P_i to P_j if and only if $a_{i,j} \neq 0$. We denote the arc from P_i to P_j using the notation (P_i, P_j).

Example. Say we have the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & -3 \\ 1 & 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 4i & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 & 7 \\ -5 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we get the corresponding graphs for each matrix:

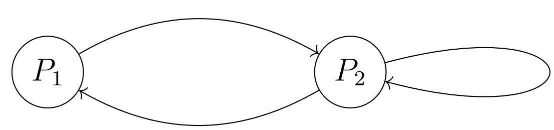
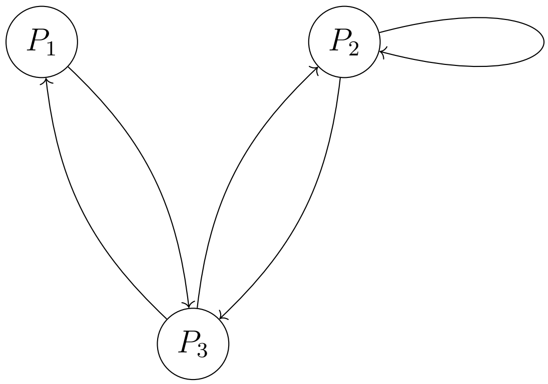


Figure 3. Graph corresponding to matrix A

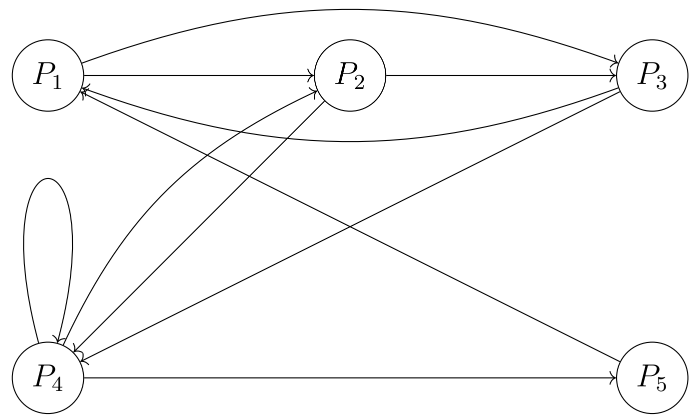
Definition 5.6. A directed path γ is a sequence of arcs $(P_{i_1}, P_{i_2}), (P_{i_2}, P_{i_3}), \dots$ in Γ. The length of a directed path is the number of arcs in it if it is finite, or ∞ if it is not. A cycle is a path that begins and ends at the same node, this node must appear twice and all other nodes must appear only once. A cycle of length 1 is a loop.

Definition 5.7. A directed graph Γ is strongly connected if between each pair of distinct nodes P_i, P_j, there is a directed path that begins in P_i and ends in P_j.

Theorem 5.15. *Let $A \in M_n$. Then A has SC property if and only if $\Gamma(A)$ is strongly connected.*



Graph corresponding to matrix B



Graph corresponding to matrix C

Proof. If $A \in M_n$ has property SC, then property SC tell us that for any $p, q \in \{1, 2, \ldots, n\}$ there is a sequence $k_1 = p, k_2, \ldots, k_m = q$ such that all elements $a_{k_1,k_2}, a_{k_2,k_3}, \ldots, _{k_{m-1},k_m}$ are nonzero. With the definition of $\Gamma(A)$, then this is equivalent to there being a path between all pairs of nodes P_p and P_q, and thus $\Gamma(A)$ is strongly connected.
A similar rationale can be used to prove that if $\Gamma(A)$ is strongly connected, then A has property SC, and as such we reach our conclusion. ∎

To see if $A \in M_n$ has SC, one may simply look $\Gamma(A)$. For smaller n's, you can check by looking at it. For larger n's, using computational algorithms is a more viable option.

Theorem 5.16. *Let $A \in M_n$ and let P_i and P_j to be nodes in $\Gamma(A)$. The following are equivalent.*

(a) *There is a directed path of length m between P_i and P_j.*
(b) *The i, j entry of $|A|^m$ is nonzero.*
(c) *The i, j entry of $M(A)^m$ is nonzero.*

Proof. We prove this by induction. For $m = 1$, it is obvious. For $m = 2$ we have

$$(|A|^2)_{i,j} = \sum_{k=1}^{n} |A|_{i,k} \, |A|_{k,j} = \sum_{k=1}^{n} |a_{i,k}| \, |a_{k,j}|$$

. Therefore, $|A|^2_{i,j} \neq 0$ if and only if there is at least one element of the sum that is nonzero. But this is equivalent to there being a path of length 2 between P_i and P_j.
Suppose we have proved the assertion for $m = q$. Then

$$(|A|^{q+1})_{i,j} = \sum_{k=1}^{n} |A|^q_{i,k} \, |A|_{k,j} = \sum_{k=1}^{n} |A|^q_{i,k} \, |a_{k,j}|$$

. But an element $|A|^{q+1}$ is nonzero if and only if there is a $k \in 1, 2, \ldots, n$ such that $|A|^q_{i,k} \neq 0$ and $|a|_{k,j} \neq 0$. But this implies there is a path of length $q + 1$ between P_i and P_j, and by induction, we get conclusion (b). Using the same argument for $M(A)$ we conclude (c). ∎

Definition 5.8. Let $A \in M_n$. We say that A is nonnegative if every entry $a_{i,j}$ is real and nonnegative. We say that A is positive if every entry $a_{i,j}$ is real and positive. We denote them using $A \geq 0$ and $A > 0$ respectively.

Corollary 5.17. *Let $A \in M_n$. Then $|A|^m > 0$ if and only if there is a directed path of length m from every node P_i to every other node P_j. The same is true for $M(A)$.*

Proof. The result is derived easily from theorem 5.16. ∎

Corollary 5.18. *Let $A \in M_n$. The following are equivalent:*

(*a*) *A has property SC.*
(*b*) *$(I_n + |A|)^{n-1} > 0$.*
(*c*) *$(I_n + M(A))^{n-1} > 0$.*

Proof. As I_n and $|A|$ commute (as in, $I_n |A| = |A| I_n$),we can apply the binomial expansion to $(I_n - |A|)^{n-1} = I_n + (n-1)|A| + \binom{n-1}{2}|A|^2 + \cdots + |A|^{n-1} > 0$ if and only if for each pair i, j at least one of the matrices $|A|, \dots, |A|^{n-1}$ has a positive entry in position i, j. But theorem 5.16 ensures this happens if and only if there is a directed path in $\Gamma(A)$ from P_i to P_j for all pairs i, j. Thus, $\Gamma(A)$ is strongly connected. By theorem 5.15, that is equivalent to A having property SC.
The same argument works for $M(A)$, and thus we reach our conclusion. ∎

We shall introduce one more notion that will help us characterize property SC, that being the notion of irreducibility.

Definition 5.9. A matrix $P \in M_n$ is a permutation matrix if in each row and each column, there is only one entry of 1 and the rest are 0.

There is a correspondence between permutations and permutation matrices. To be general, for a permutation matrix $P = [p_{i,j}] \in M_n$, $p_{i,j} = 1$ if and only if $j = \tau(i), \tau \in S_n$.

Remark 5.19. The transpose of a permutation matrix is also a permutation matrix.

Remark 5.20. Multiplying a matrix $A \in M_n$ with a permutation matrix $P \in M_n$ only permutes the rows and columns of the matrix, and does not modify the value of the entries of A.

Remark 5.21. The transpose of a permutation matrix is its own inverse.

Proof. Suppose we have a permutation matrix $P \in M_n$ and its associated permutation $\tau \in S_n$. The inverse of a the permutation τ^{-1} corresponds to permutation matrix $Q = [q_{i,j}] \in M_n$ such that $q_{i,j} = 1$ if and only if $j = \tau^{-1}(i)$ which is equivalent to $\tau(i) = j$, which is the transpose of P_1. As such, the permutation matrix corresponding to τ^{-1} is P^T and the conclusion is trivial. ∎

Example. Say we have the permutation $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$. Its corresponding permutation matrix is $P_\tau = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Definition 5.10. A matrix $A \in M_n$ is reducible if there is a permutation matrix $P \in M_n$ such that

$$P^T A P = \left[\begin{array}{c|c} B & C \\ \hline \mathbf{0}_{n-r,r} & D \end{array}\right] \text{ and } 1 \leq r \leq n-1$$

where B, C, D are blocks of size at least 1×1 and $\mathbf{0}_{n-r,r}$ is a block of size $n - r \times r$ with all elements 0.

Definition 5.11. The matrix $A \in M_n$ is irreducible if it is not reducible.

Theorem 5.22. *Let $A \in M_n$. Then the following are equivalent.*

(a) *A is irreducible.*
(b) $(I_n + |A|)^{n-1} > 0$.
(c) $(I_n + M(A))^{n-1} > 0$.

Proof. To show that (a) and (b) are equivalent, it is enough to prove that A is reducible if and only if $(I_n+|A|)^{n-1}$ has a zero entry. Let's suppose that A is reducible and that for some permutation matrix P we have $P^TAP = \left[\begin{array}{c|c} B & C \\ \hline \mathbf{0}_{n-r,r} & D \end{array}\right] = \tilde{A}$, in which $B \in M_r$, $D \in M_{n-r}$ and $1 \leq r \leq n-1$. We notice that $P^T |A| P = \left|P^TAP\right| = \left|\tilde{A}\right|$ since P does not modify the value of the entries of A. We further notice that each of the matrices $\left|\tilde{A}\right|^2, \left|\tilde{A}\right|^3, \ldots, \left|\tilde{A}\right|^{n-1}$ has lower left block $\mathbf{0}_{n-r,r}$. Thus

$$\begin{aligned} P^T(I_n + |A|)^{n-1}P &= (I_n + P^T |A| P)^{n-1} = (I_n + \left|\tilde{A}\right|)^{n-1} \\ &= I_n + (n-1)\left|\tilde{A}\right| + \binom{n-1}{2}\left|\tilde{A}\right|^2 + \cdots + \left|\tilde{A}\right|^{n-1} \end{aligned}$$

in which each summand has a lower left $\mathbf{0}_{n-r,r}$ block. As such, $(I_n + |A|)^{n-1}$ is reducible, so it has a zero entry. Suppose we take two indices p, q such that $p \neq q$ and p, q entry of $(I_n + |A|)^{n-1}$ is zero. That implies there is no directed path between from the node P_p to the node P_q. Denote the set of nodes P_i such that there is a directed path in $\Gamma(A)$ from P_i to P_q as S_1 and the set of all other nodes as S_2, clearly, $S_1 \bigcup S_2 = \{P_1, P_2, \ldots, P_n\}$ and $P_q \in S_1 \neq \varnothing$. Therefore, $S_2 \neq \{P_1, P_2, \ldots, P_n\}$. If there were a path from any node in $P_i \in S_2$ to any node in S_1, there would be by definition a path between $P_i and P_q$. As such, there are no paths from any node in S_2 to any node in S_1. Now, relabeling the nodes such that $S_1 = \{\tilde{P_1}, \ldots, \tilde{P_r}\}$ and $S_2 = \{\tilde{P_{r+1}}, \ldots, \tilde{P_n}\}$ and let P be the permutation matrix corresponding to that relabeling, then:

$$\tilde{A} = P^TAP = \left[\begin{array}{c|c} B & C \\ \hline \mathbf{0}_{n-r,r} & D \end{array}\right], \; B \in M_r, \; D \in M_{n-r}$$

and therefore A is irreducible. The same argument works for M(A) and we have reached our conclusion. ∎

All properties and interpretations of property SC can be summed up in one theorem.

Theorem 5.23. *Let $A \in M_n$. Then the following are equivalent:*

(a) *A has property SC.*
(b) *$\Gamma(A)$ is strongly connected.*
(c) $(I_n + |A|)^{n-1} > 0$.
(d) $(I_n + M(A))^{n-1} > 0$
(e) *A is irreducible.*

Proof. The result is trivially derived as a consequence of theorem 5.15, corollary 5.18, and theorem 5.22. ∎

6. Eigenvalue perturbation theorems

Let $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \in M_n$ be a matrix, where *diag* denotes a matrix with all off-diagonal elements 0 and the i-th diagonal entry equal to the i-th entry in the sequence. Let $E = [e_{i,j}] \in M_n$. Looking at the perturbed matrix $D + E$ and applying theorem 5.1 to it, we get the following result

$$\bigcup_{i=1}^{n} \{z \in \mathbb{C} \mid \ |z - \lambda_i - e_{i,i}| \le R_i(E)\}$$

which is contained in the set

$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} \mid \ |z - \lambda_i| \le \sum_{j=1}^{n} |e_{i,j}| \right\}.$$

Thus, if $\hat{\lambda}$ is an eigenvalue of $D + E$ then there is some eigenvalue λ_i of D such that $\left|\hat{\lambda} - \lambda_i\right| \le |||E|||_\infty$.

Observation 6.1. Let $A \in M_n$ be diagonalizable and suppose that $A = S\Lambda S^{-1}$, in which S is nonsingular and Λ is diagonal. Let $E \in M_n$, and consider the perturbed matrix $A+E$. If $\hat{\lambda}$ is an eigenvalue $A + E$, then there is an eigenvalue λ of A such that

$$\left|\hat{\lambda} - \lambda\right| \le |||S|||_\infty \left|\left|\left|S^{-1}\right|\right|\right|_\infty |||E||| = \kappa_\infty(S)\, |||E|||_\infty$$

in which κ_∞ is the condition number with respect to them $|||\cdot|||_\infty$

Proof. Since $A + E$ has the same eigenvalues as $S^{-1}(A + E)S = S^{-1}(S\Lambda S^{-1} + E)S = \Lambda + S^{-1}ES$ and since Λ is diagonal, we can apply the previous argument and we reach that $\left|\hat{\lambda} - \lambda\right| \le |||S^{-1}ES|||$ and applying property (4) of matrix norms we get our conclusion. ∎

While this observation gives us a nice bound for the perturbation of the eigenvalues, we can generalize our previous observation in the following theorem.

Theorem 6.2 (Bauer–Fike theorem)**.** *[BF60] Let $A \in M_n$ be a diagonalizable matrix and suppose that $A = S\Lambda S^{-1}$, in which S is nonsingular and Λ is diagonal. Let $E \in M_n$ and $|||\cdot|||$ that is induced by an absolute norm on $\mathbb{C}^n$. If $\hat{\lambda}$ is an eigenvalue of $A + E$, then there is an eigenvalue λ of A such that*

$$\left|\hat{\lambda} - \lambda\right| \le |||S||| \left|\left|\left|S^{-1}\right|\right|\right| |||E||| = \kappa(S)\, |||E|||$$

in which $\kappa(\cdot)$ is the condition number with respect to the matrix norm $|||\cdot|||$.

Proof. If $\hat{\lambda}$ is an eigenvalue of $A + E$, then it s also an eigenvalue of $S^{-1}(A + E)S = \Lambda + S^{-1}ES$ then $\hat{\lambda}I_n - \Lambda - S^{-1}ES$ is singular. If $\hat{\lambda}$ is also an eigenvalue of A, then the conclusion is obvious. If $\hat{\lambda}$ is not an eigenvalue of A, then $\hat{\lambda}I_n - A$ is nonsingular. $(\hat{\lambda}I_n - A)^{-1}(\hat{\lambda}I_n - \Lambda - S^{-1}ES) = I_n - (\hat{\lambda}I_n - A)^{-1}S^{-1}1ES$ is singular. By observation 4.4, that implies $\left|\left|\left|(\hat{\lambda}I_n - A)^{-1}S^{-1}1ES\right|\right|\right| \geq 1$

$$\begin{aligned}1 &\leq \left|\left|\left|(\hat{\lambda}I_n - A)^{-1}S^{-1}1ES\right|\right|\right| \leq |||S^{-1}ES||| \left|\left|\left|(\hat{\lambda}I_n - \Lambda)^{-1}\right|\right|\right| \\ &= |||S^{-1}ES||| \max_{1\leq i\leq n} \left|\hat{\lambda} - \lambda_i\right|^{-1} = \frac{|||S^{-1}ES|||}{\min_{1\leq i\leq n}\left|\hat{\lambda} - \lambda_i\right|}\end{aligned}$$

and due since $\hat{\lambda}$ is not an eigenvalue of A, we can divide by $\min_{1\leq i\leq n}\left|\hat{\lambda} - \lambda_i\right|$ and we get

$$\min_{1\leq i\leq n}\left|\hat{\lambda} - \lambda_i\right| \leq \left|\left|\left|S^{-1}ES\right|\right|\right| \leq \left|\left|\left|S^{-1}\right|\right|\right| \, |||S||| \, |||E||| = \kappa(S) \, |||E|||$$

■

Remark 6.3. All the previous matrix norms we have defined in this paper are induced by an absolute norm on $\mathbb{C}^n$. What this notion means, more rigorously, is outside the scope of this paper. If you want to learn more, refer to chapter 5 of [HJ13].

The preceding theorem can also be written as

$$\frac{\left|\hat{\lambda} - \lambda\right|}{|||E|||} \leq \kappa(S).$$

Say we compute the eigenvalue $\hat{\lambda}$ of the perturbed matrix $A + E$ and we want to use the above relation to estimate an eigenvalue λ of A. Then we notice that if $\kappa(S)$ is small, especially if it's near 1 or lower, then the changes between λ and $\hat{\lambda}$ are relatively small. But if $\kappa(S)$ is large, this estimation is likewise broad and therefore poor.
Investigating the perturbation bound of certain kinds of matrices can yield some interesting results.

Definition 6.1. A matrix $A \in M_n$ is normal if $AA^* = A^*A$, as in, it commutes with its own conjugate transpose.

Theorem 6.4. *If U is a unitary matrix, then the sum of the squares of the absolute value of each element of each row and column is 1.*

Proof. Let us denote the $[u_{i,j}]$ as the entries of U. From $UU^* = I_n$ we get that for any $i \neq j$, $\sum_{k=1}^{n} u_{k,i}\overline{u_{k,j}} = 0$ and that for all $i = j$, the row sum $\sum_{k=1}^{n} u_{k,i}\overline{u_{k,i}} = \sum_{k=1}^{n} |u_{k,i}|^2 = 1$.
A similar argument shows us that $\sum_{k=1}^{n} |u_{j,k}|^2 = 1$. ■

Theorem 6.5. *If $A \in M_n$ is a normal matrix if and only if there exists a unitary matrix $U \in M_n$ such that $A = U\Lambda U^*$ where Λ is a diagonal matrix.*

The proof for this theorem is unfortunately outside the scope of this paper. If you want to learn more, refer to Chapter 2 of [HJ13].
Using the preceding theorem, we get the following corollary.

Corollary 6.6. *Let $A, E \in M_n$, and suppose that A is normal. If $\hat{\lambda}$ is an eigenvalue of $A + E$ then there is an eigenvalue λ of A such that $\left|\hat{\lambda} - \lambda\right| \leq |||E|||_F$.*

Proof. Using the decomposition $A = U\Lambda U^*$ where $U \in M_n$ is unitary and the Frobenius norm in 6.2 and due to the unitary invariance of the Frobenius norm we get

$$\left|\hat{\lambda} - \lambda\right| \leq |||U^*EU|||_F = |||E|||_F .$$

■

Theorem 6.7 (Hoffman and Wielandt). *[HW53] Let $A, E \in M_n$, such that A and $A+E$ are both normal. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A in some given order and $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n$ be the eigenvalues of $A+E$ in some given order. Then there is a permutation $\tau(\cdot) \in S_n$ such that*

$$\sum_{i=1}^{n} \left|\hat{\lambda}_{\tau(i)} - \lambda_i\right|^2 \leq |||E|||_F^2 .$$

Proof. Let $V, W \in M_n$ be the unitary matrices such that $A = V\Lambda V^*$ and $A + E = W\hat{\Lambda}W^*$ where Λ and $\hat{\Lambda}$ are the diagonal matrices containing the eigenvalues of A and $A+E$ respectively and let $U = V^*W = [u_{i,j}]$. Then

$$\begin{aligned}
|||E|||_F^2 = |||A + E - A|||_F^2 &= \left|\left|\left|W\hat{\Lambda}W^* - V\Lambda V^*\right|\right|\right|_F^2 \\
&= \left|\left|\left|V^*W\hat{\Lambda} - \Lambda V^*W\right|\right|\right|_F^2 = \left|\left|\left|U\hat{\Lambda} - \Lambda U\right|\right|\right|_F^2 \\
&= \sum_{i,j=1}^{n} \left|\hat{\lambda}_i - \lambda_j\right|^2 |u_{i,j}|^2 .
\end{aligned}$$

As $U^* = W^*V$ which means that U is also unitary, and as such the sum of the squares of the absolute values of each element of each row and column is 1. Let $S \in M_n$ be a class of matrices, not necessarily unitary, such that the sum of the squares of the absolute values of each element of each row and column is 1. Therefore

$$\begin{aligned}
|||E|||_F^2 &= \sum_{i,j=1}^{n} \left|\hat{\lambda}_i - \lambda_j\right|^2 |u_{i,j}|^2 \\
&\geq \min\{\sum_{i,j=1}^{n} \left|\hat{\lambda}_i - \lambda_j\right|^2 s_{i,j} \ : S = [s_{i,j}]\}.
\end{aligned}$$

If we define the function $f(S) = \sum_{i,j=1}^{n} \left|\hat{\lambda}_i - \lambda_j\right|^2 s_{i,j}$, we can observe that is a linear function on the set of matrices of class S. Therefore, f attains its minimum for a permutation matrix $P = [p_{i,j}]$. If P^T corresponds to the permutation $\tau \in S_n$, then

$$|||E|||_F^2 \geq \sum_{i,j=1}^{n} \left|\hat{\lambda}_i - \lambda_j\right|^2 p_{i,j} = \sum_{i=1}^{n} \left|\hat{\lambda}_{\tau(i)} - \lambda_i\right| .$$

■

For our last theorem, we shall refer back to diagonalizable matrices.

Theorem 6.8. *Let $A \in M_n$ be a diagonalizable matrix with $A = S\Lambda S^{-1}$ where Λ is a diagonal matrix. Let $|||\cdot|||$ be a matrix norm on M_n that is induced by an absolute vector norm $\|\cdot\|$ on $\mathbb{C}^n$. Let $\hat{x} \in C^n$ be a nonzero vector, let $\hat{\lambda}$ and let $r = A\hat{x} - \hat{\lambda}\hat{x}$. Then* (*a*) *There is an eigenvalue* λ *of* A *such that*

$$\left|\hat{\lambda}-\lambda\right| \leq |||S||| \, \left|\left|\left|S^{-}1\right|\right|\right| \frac{\|r\|}{\|\hat{x}\|} = \kappa(S)\frac{\|r\|}{\|\hat{x}\|}.$$

in which $\kappa(\cdot)$ *is the condition number with respect to norm* $|||\cdot|||$.

(*b*) *If* A *is normal, then there is an eigenvalue of* A *such that*

$$\left|\hat{\lambda}-\lambda\right| \leq \frac{\|r\|_2}{\|\hat{x}\|_2}.$$

Proof. The bound is easily shown if $\hat{\lambda}$ is an eigenvalue of A. If $\hat{\lambda}$ is not an eigenvalue of A, then $r = A\hat{x} - \hat{\lambda}\hat{x} = S(\Lambda - \hat{\lambda}I_n)S^{-1}\hat{x}$ and $\hat{x} = S(\Lambda - \hat{\lambda}I_n)^{-1}S^{-1}r$. Therefore

$$\begin{aligned}\|\hat{x}\| &= \left\|S(\Lambda-\hat{\lambda}I_n)^{-1}S^{-1}r\right\| \leq \left|\left|\left|S(\Lambda-\hat{\lambda}I_n)^{-1}S^{-1}\right|\right|\right| \|r\| \\ &\leq |||S||| \, \left|\left|\left|S^{-1}\right|\right|\right| \, \left|\left|\left|(\Lambda-\hat{\lambda}-I_n)^{-1}\right|\right|\right| \|r\| = \kappa(S) \left|\left|\left|(\Lambda-\hat{\lambda}-I_n)^{-1}\right|\right|\right| \|r\| \\ &= \kappa(S) \max_{\lambda\in\sigma(A)} \left|\lambda-\hat{\lambda}\right|^{-1} \|r\|.\end{aligned}$$

And as such, we reach that

$$\|\hat{x}\| \min_{\lambda\in\sigma(A)} \left|\lambda-\hat{\lambda}\right| \leq \kappa(A)\,\|r\| \Leftrightarrow \min_{\lambda\in\sigma(A)} \left|\lambda-\hat{\lambda}\right| \leq \kappa(A)\frac{\|r\|}{\|\hat{x}\|}$$

We conclude (b) by using the same method with decomposition $A = U\Lambda U^*$. ■

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Further reading

For a deeper dive into prerequisite knowledge, [HJ13] is an excellent textbook for the subject of matrix analysis. For a book more centered on the Gershgorin circle theorem(5.1) and other similar eigenvalue inclusion sets, I point you towards [Var04].

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Email address: papuseldragos@gmail.com