How many ways can we tile a Rectangular Grid with Dominos?

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Euler Circle

July 14, 2024

Does a tiling exist?

Theorem 0.1

Let G be an $m \times n$ grid. Then, there is a domino tiling if and only if mn is even.

Recursions for $2 \times n$ grids

Definition 0.2

The n^{th} Fibonacci number, $F(n)$, is defined as the sum of the two previous Fibonacci numbers, $F(n-1)$ and $F(n-2)$, where $F(1) = F(2) = 1$.

Theorem 0.3

If we are given a $2 \times n$ grid, then the number of tilings is the $(n+1)th$ Fibonacci number.

Proof with Induction

As with all proofs using induction, we have to start with the base case.

Proof with Induction

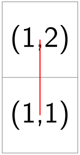


Figure 1: Our 2×1 grid

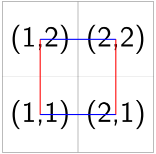


Figure 2: Our 2×2 grid

Proof with Induction

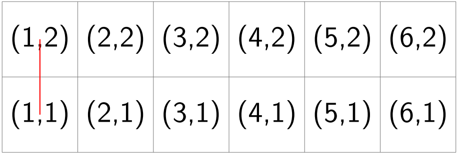


Figure 3: a $2 \times n$ grid with a vertical domino

Proof with Induction

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

Figure 4: a $2 \times n$ grid with a horizontal domino

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

Figure 5: a $2 \times n$ grid with 2 horizontal dominoes

Proof with Induction

We now have a recursion with $F(n+1) = F(n) + F(n-1)$. As a result, the number of tilings for a $2 \times n$ grid is the $(n+1)^{th}$ Fibonacci number.

Adjacency Matrices, Perfect Matchings, and Bipartite graphs

If we label each square in a grid as shown below, then we notice a very interesting thing.

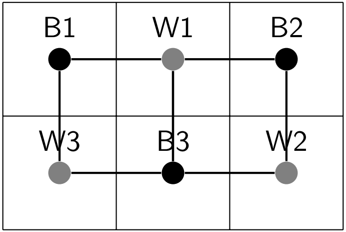


Figure 6: Our 2×3 grid with alternating labels

Adjacency Matrices, Perfect Matchings, and Bipartite Graphs

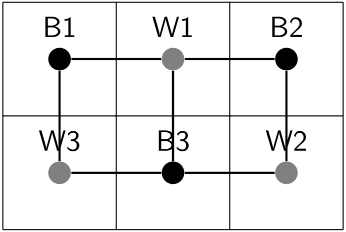


Figure 7: Our 2×3 grid with alternating labels

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Adjacency Matrices, Perfect Matchings, and Bipartite Graphs

Take our matrix from earlier. If we look at our matrix, there is a very easy way to keep track of configurations. If we take a permutation of the white vertices and connect it to the black vertex of its position, then we have an effective way to count the configurations.

(Kasteleyn) Signings

Definition 0.4

A signing of G is weighting each edge with a 1 or -1. If $\sigma : E(G) \to (-1, 1)$, then A^σ, our signed adjacency matrix, is given by assigning each a_{ij} a value. Our new a_{ij}, a_{ij}^σ, is given by the following piecewise function:

$$a_{ij}^\sigma = \begin{cases} \sigma & (b_i, w_j) \text{ is an edge} \\ 0 & \text{Otherwise} \end{cases}$$

Results

So we'll first draw our grid and label each square with a black or white vertex. Then, we'll draw a smaller grid around it.

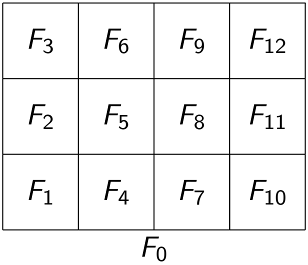


Figure 8: Our 3×4 grid with faces

Results

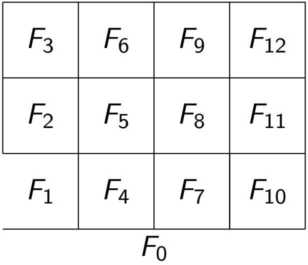


Figure 9:

Results

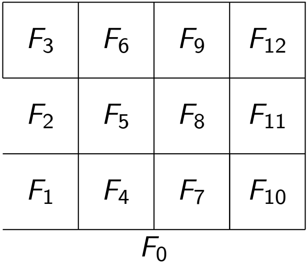


Figure 10:

Results

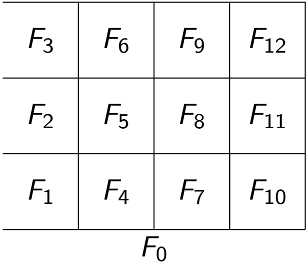


Figure 11:

Results

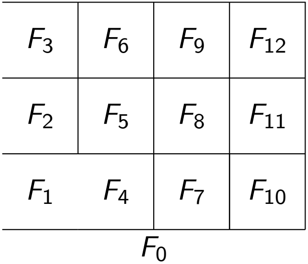


Figure 12:

Results

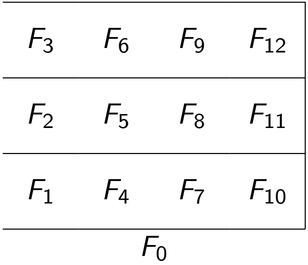
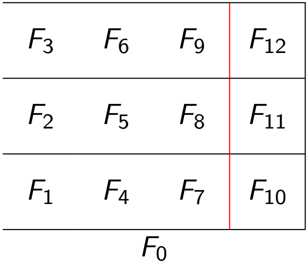
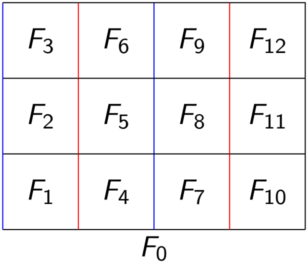


Figure 13:

Results



Results



Results

Our matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

with a determinant of 90.