HOW MANY WAYS ARE THERE TO TILE A REGULAR RECTANGULAR GRID WITH DOMINOES?

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Abstract. In this paper, we will first see if a tiling exists. Then, we will attempt to calculate the number of tilings with recursions. After that, we will use linear algebra. Our main theorem will be to change the matrix of the permanent such the determinant of the new matrix is the same as the permanent. Finally, we will apply the theorem in a way such that we can calculate the number of tilings in polynomial-time.

1. Introduction

To get the number of domino tilings, what you'll do is remove certain edges from the grid. Then, you'll put them back in but assign them weights of 1 or -1. After that, we create an adjacency matrix while taking into account our weights. Finally, we take the determinant of our matrix to get the number of tilings. But before we get to the result, we will start by determining whether a configuration exists. We will then check if a recursion for a $2 \times n$ grid exists. Once we see how easy it is, we look at larger examples. However, we soon find that the larger our dimensions, the harder it becomes. We will then create an adjacency matrix. After that, we will find that the number of tilings is just the permanent of our adjacency matrix. However, we find that permanents cannot be solved in polynomial-time so we introduce Kasteleyn signings. Kasteleyn signings allow us to change some of the 1's to -1's so that our determinant of the new matrix is the same as the permanent of our original matrix. We will use our lemmas with Euler's polyhedron formula, $V + F = E + 2$, to efficiently create a Kasteleyn signing. Our pattern will have us remove certain edges and then put them back in with their weights. Finally, we take the determinant of our new adjacency matrix.

2. Preliminaries

Definition 2.1. A domino is a a rectangle formed by two adjacent squares.

Definition 2.2. A domino tiling is a combination of disjoint dominoes such that the whole grid is covered by dominoes and there are no dominoes sticking out of the grid.

Definition 2.3. A bipartite graph is a graph whose vertices can be separated into two parts, so that each edge only goes between both parts.

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Definition 2.4. A perfect matching in a graph is a selection of edges that covers each vertex exactly once.

Definition 2.5. A cycle in G is a sequence of adjacent vertices and edges that return to the same vertex.

Definition 2.6. A cycle, C, is evenly-placed if G has a perfect matching excluding all the edges and vertices of C.

Definition 2.7. A signing of G is weighting each edge with a 1 or -1. If $\sigma : E(G) \to (-1, 1)$, then A^σ, our signed adjacency matrix, is given by assigning each a_{ij} a value. Our new a_{ij}, a^σ_{ij}, is given by the following piecewise function:

$$a^\sigma_{ij} = \begin{cases} \sigma & (b_i, w_j) \text{ is an edge} \\ 0 & \text{Otherwise} \end{cases}$$

Definition 2.8. Given σ on G, a cycle, C, is properly-signed if its length matches the weight of the edges appropriately: If $|C|$, the length of the cycle, equals $2l$, then the number of negative edges on C, n_C, will have the opposite parity of l, i.e. $n_C \equiv l - 1 \pmod 2$

Definition 2.9. The permanent of a square matrix, denoted as $per(A)$, is expressed as

$$\sum_{\pi \in S_N} a_{1,\pi(1)} ... a_{N,\pi(N)}.$$

Definition 2.10. The determinant of a square matrix, denoted as $det(A)$, is defined by

$$\sum_{\pi \in S_N} sgn(\pi) a_{1,\pi(1)} ... a_{N,\pi(N)},$$

where $sgn(\pi)$ is defined by how many transpositions take place.

Definition 2.11. A Kasteleyn signing is a signing of G, such that

$$per(A) = |det(A^\sigma)|.$$

3. Does a tiling exist?

Before we check how many ways there are to tile an $m \times n$ grid, where m and n are integers, we must check if a configuration exists.

Theorem 3.1. *Let G be an $m \times n$ grid. Then, there is a domino tiling if and only if mn is even.*

Proof. Given a domino tiling, we know each domino occupies 2 squares. If we assume that N dominoes were used, then $2N$ squares were used. In other words, $2N = mn$. Thus, if mn is even, that implies m or n is even. Without loss of generality, let m be even and written as $m = 2k$ where $k \in \mathbb{N}$. We can form a domino tiling with dominoes that are upright as shown below.

We know that each column will fit exactly k dominoes since each dominoes takes up two rows and we have exactly $2k$ rows in a column. ∎

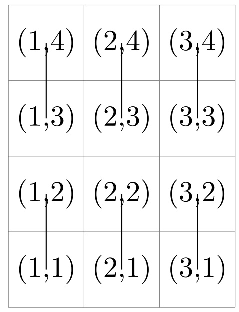


Figure 1. a 4×3 grid

Thus, if our area of the grid is even, we have at least one possible domino tiling.

4. A simpler example of an $m \times n$ grid

Before we look at larger examples of $m \times n$ grids, we will first look at examples where m is 2.

Definition 4.1. The n^{th} Fibonacci number, $F(n)$, is defined as the sum of the two previous Fibonacci numbers, $F(n-1)$ and $F(n-2)$, where $F(1) = F(2) = 1$.

Theorem 4.2. *If we are given a $2 \times n$ grid, then the number of tilings is the $(n+1)th$ Fibonacci number.*

Proof. Using induction, we start with the base case. We first start with the number of configurations for $n = 1$ & 2. We find that there is 1 configuration for a 2×1 grid and 2 configurations for a 2×2 grid, which are the 2^{nd} and 3^{rd} Fibonacci numbers, respectively. We define a function, $F(n+1)$, that gives us the number of configurations for the $2 \times n$ grid. Then, we have two possibilities: either the first domino is laid upright or horizontally. Let us explore the former.

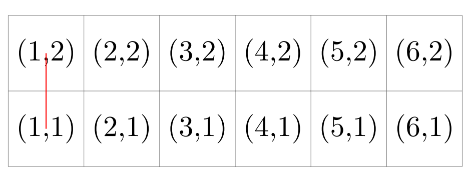


Figure 2. a $2 \times n$ grid with a vertical domino

If we lay the domino upright, we get a recursion for a $2 \times (n-1)$ grid.

If we lay down the domino horizontally, we get a different recursion.

No matter what, we will have to place a domino from $(1,2)$ to $(2,2)$, as shown in figure 4 since there is no other way to reach $(1,2)$.

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

Figure 3. a $2 \times n$ grid with a horizontal domino

(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

Figure 4. a $2 \times n$ grid with 2 horizontal dominoes

Thus, we now get a recursion for a $2 \times (n-2)$ grid, or $F(n-1)$. In other words, we find that $F(n+1) = F(n) + F(n-1)$. Thus, the number of tilings for a $2 \times n$ grid is $F(n+1)$, the $(n+1)^{th}$ Fibonacci number. ■

Now we look at larger examples. Can we do something similar to the $2 \times n$ grid? We could, but it would just get really complicated. For example, even with a small 3×4 grid, it is incredibly difficult. Do we fill the first row with horizontal dominoes to get a 2×4 grid? What about a 2×3 grid? How many ways can we form the 2×3 grid? Even with such a small grid, counting with recursion is incredibly difficult. Thus, we look for an easier way.

5. Perfect Matchings, Bipartite Graphs, and Adjacency Matrices

If we put a vertex in the middle of each square, such that we alternate between black and white labels like a chessboard, as shown below, we find that each domino is in fact, an edge between a black and white vertex.

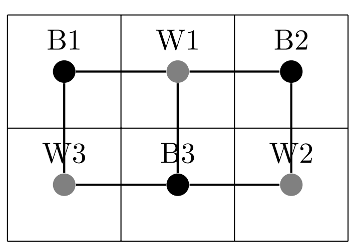


Figure 5. Our 2×3 grid with alternating labels

We can easily realize that a configuration is just a perfect matching between the black and white vertices.

Now, we create an adjacency matrix, A, such that it tells us which points are connected. As a reminder, the element in the i^{th} row and j^{th} row, a_{ij}, is a 1 if B_i and W_j are an edge and a 0 if they are not. In other words, if B_i and W_j is an element of E(G), the set of all edges in G, then $a_{ij} = 1$. For example, with our 2×3 grid, B_1 is connected to W_1 and W_3 so our first row would be:

$$1 \quad 0 \quad 1.$$

Similarly, our second row would be

$$1 \quad 1 \quad 0.$$

Finally, our whole matrix would be

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Our adjacency matrix would be an $N \times N$ matrix since we have N black vertices and N white vertices. The purpose of the adjacency matrix is that we can tell if a configuration exists by checking certain a_{ij}'s. For example, from our 2×3 grid or 3×3 matrix, we can see that if we check the positions, a_{11}, a_{22}, and a_{33}, we get a configuration. We find that all tilings are just a permutation of N. We are just taking the permutation of the white vertices and mapping it to the black vertices of its position. For example, the permutation,

$$3 \quad 1 \quad 2,$$

is mapping B_1 to W_3, B_2 to W_1, and B_3 to W_2.

This permutation, is in fact, a tiling since a_{13}, a_{21}, and a_{32} all give a 1. Unfortunately, not all permutations give us a configuration. For example, the permutation,

$$2 \quad 3 \quad 1,$$

is a configuration that simply doesn't exist. This is because we cannot connect B_1 to W_2 and B_2 to W_3 with a normal domino. Thus, if we create a permutation of the white vertices, π, such that we get a 1 from $a_{(i,\pi(i))}$ for all i, then we have just created a configuration.

Now, we aim to find a way to count these configurations. We notice that if we multiply each $a_{(i,\pi(i))}$ for all i, then we get a 1 or a 0 . If we get a 1, then that means our perfect matching does, in fact, exist when we actually lay the dominoes. However, if we get a 0, that works out perfectly since that means this permutation will not contribute to our total sum.

Thus, our total number of tilings, $T(m, n)$, equals

$$\sum_{\pi \in S_N} a_{1,\pi(1)} \dots a_{N,\pi(N)}$$

If you have studied linear algebra before, you might notice a very similar resemblance. This formula is exactly the same as the permanent of a matrix, denoted as $per(A)$. So, we can find the number of configurations by writing out the adjacency matrix and then calculating the permanent. However, we face a very annoying problem.

6. NP and #P problems

Definition 6.1. An answer is calculated in polynomial-time if the running time of the algorithm is proportional to the input size, i.e. a larger grid will take a proportionally larger amount of time to solve.

Definition 6.2. NP problems are decision problems. In other words, they are problems whose answers can be evaluated in polynomial-time.

Definition 6.3. #P problems are the counting versions of NP problems.

Definition 6.4. Problems are #P-Complete if they are in #P and all other problems in #P can be reduced to it in polynomial-time.

The problem with calculating $per(A)$ is that permanents are hard to calculate. They can't be calculated in polynomial-time. Sure, we can calculate it, but it just takes so much time! We look for a way to express $per(A)$ in a better way that can be calculated in polynomial-time.

Definition 6.5. The sign function, denoted as $sgn(x)$, gives an output of ± 1 depending on the parity of x.

As a reminder, the determinant of a square matrix, $det(A)$, is defined by

$$\sum_{\pi \in S_N} sgn(\pi) a_{1,\pi(1)} \ldots a_{N,\pi(N)},$$

where $sgn(\pi)$ is defined by how many transpositions take place. Of course, it is hard to account for the alternating sign, but there is a very valuable property of determinants: they can be calculated in polynomial-time.

Thus, our goal is to now create a matrix, A', such that

$$per(A) = |det(A')|,$$

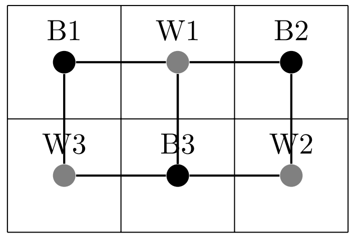
where $det(A)$ denotes the determinant. We can do so by weighing each edge in a certain way.

7. Kasteleyn Signings

Thus, we want to create a Kasteleyn signing. Let us try and make a Kasteleyn signing of a 2×3 grid.

Given the grid below, we have the matrix,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$



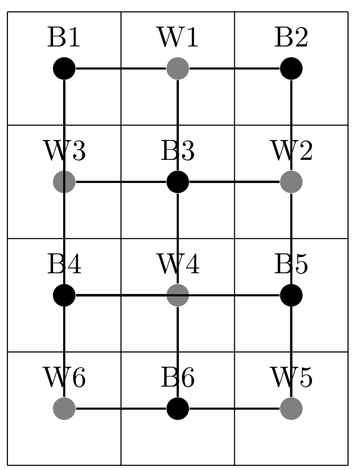
Now, we want to change some of the 1's to a -1 such that the determinant of our signing is 3. The current determinant is 1+0+0=1. If we change our matrix to the following,

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

we get that

$$|det(A^{\sigma})| = 3.$$

Let us try another example with a 4×3 grid.



Our adjacency matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

We find that the permanent is 11. The following matrix has a determinant of 11.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}.$$

The method that will be derived allows us to quickly change the matrix. OK, so now we've seen that a Kasteleyn signing exists for a normal rectangular grid. However, a matrix with all 1's, as shown below, does not have a Kasteleyn signing.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Note that a normal rectangular grid cannot have such an adjacency matrix. Each vertex would have to have 3 possible edges, but this is simply not possible! For this matrix, $per(A) = 6$. Now we aim to prove that it is not possible to get a σ such that $|det(A^\sigma)| = 6$.

Proof. Without loss of generality, assume that $det(A^\sigma) = 6$, and that $a_{1,1}$=1. If we change the sign of $det(A^\sigma)$ or $a_{1,1}$ then we can follow very similar steps. Okay, so now, we have

$$det(A^\sigma) = a_{1,1}a_{2,2}a_{3,3}+a_{1,2}a_{2,3}a_{3,1}+a_{1,3}a_{2,1}a_{3,2}-a_{1,1}a_{2,3}a_{3,2}-a_{1,2}a_{2,1}a_{3,3}-a_{1,3}a_{2,2}a_{3,1}$$

. Thus, we know that $a_{2,2}$ and $a_{3,3}$ are either both 1 or -1. Now, if they were both 1, then we know that $a_{2,3}$, $a_{3,2}$ have different signs. Similarly, $a_{1,2}$, $a_{2,1}$ and $a_{1,3}$, $a_{3,1}$ have opposite signs. Thus, we would have 3 -1's, and 6 1's. However, this gives us a problem. Since there are an odd amount of -1's, we will get a -1 within our sum. Thus, $det(A^\sigma) \neq 6$. Now, we check the case where they are both -1. We know that $a_{2,3}$ and $a_{3,2}$ still have opposite signs. However, $a_{1,2}$, $a_{2,1}$ and $a_{1,3}$, $a_{3,1}$ now have same signs. Once again, we have an odd amount of -1's so $det(A^\sigma) \neq 6$. ■

So, we have seen examples of Kasteleyn signings. But how do we find Kasteleyn signings in polynomial-time?

8. The Main Theorem

Theorem 8.1. *All rectangular grids have a Kasteleyn signing, and there is an efficient way to find one.*

To do this, we will use many definitions from the preliminary. We know that our grids are planar, because all edges stay parallel or perpendicular to each other so they only intersect at the vertices in the middle of each square. We know that almost all rectangular grids are 2-connected, since we can just go around the "hole." Note that the only exceptions are $1 \times n$ grids since we cannot go outside of the grid. Now, we aim to find a condition for a Kasteleyn signing to exist.

8.1. **First Lemma.**

Lemma 8.2. *If every evenly-placed cycle in G is properly-signed, then σ is a Kasteleyn signing.*

In other words, if we give each edge a weight, ± 1, and if every evenly-placed cycle we find is properly-signed, then we know that σ is a Kasteleyn signing.

Before we start the proof, we need to define the sign of a perfect matching, M, as

$$sgn(M) = sgn(\pi) a^{\sigma}_{1,\pi(1)} ... a^{\sigma}_{N,\pi(N)} = sgn(\pi) \prod_{e \in M} \sigma(e).$$

Proof. If σ is a Kasteleyn signing, then we know that the sign of all perfect matchings must be the same. This is because, a perfect matching is supposed to contribute to our total amount of configurations, i.e. $\det(A^{\sigma})$. Now, we aim to show that for any perfect matchings, M, and M', then $sgn(M) = sgn(M')$.

If M and M' share the same sign, then $sgn(M)sgn(M') = 1$. We already know that

$$sgn(\pi)sgn(\pi') \prod_{e \in M} \sigma(e) \prod_{e \in M'} \sigma(e) = sgn(\pi)sgn(\pi') \prod_{e \in M \triangle M'} .$$

The triangle, or the symmetric difference, is the union of two sets, excluding their common elements, i.e the edges being shared. We can do this since if an edge is in both sets, then the product is 1. We can simplify our equation more by letting $\prod_{e \in M \triangle M'}$ be represented by $(-1)^L$ since the sign will be 1 or -1. We just have to find a formula. If we have a cycle, C, we need to show that it is evenly-placed to prove that the lemma holds.

We now know that each vertex will be a part of an edge from M, and another from M'. Thus, if take a vertex from M, u, and find the neighbor of u, v, then we have an edge. There's nothing special yet. However, if we take the neighbor of v, w, then we have two edges. Now, the special thing is that if we repeat this over and over, until we get back to u, we have just created a cycle.

However, what happens if u is the same point as w? Well, that's an easy fix. After all, it just contributes a 1 which allows us to remove the common edge.

So then, let's get back to our example of cycles. We now have 0 or more cycles of various lengths. Okay, so as shown in figures 7 and 6, we have M and M'.

If we overlay, them, we get the figure 8, where the common edges are colored in green.

If we remove them, our final figure is figure 9.

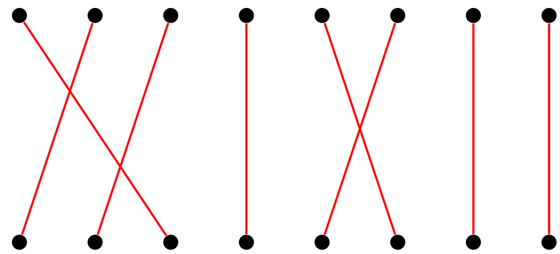


Figure 6. The perfect matching, M

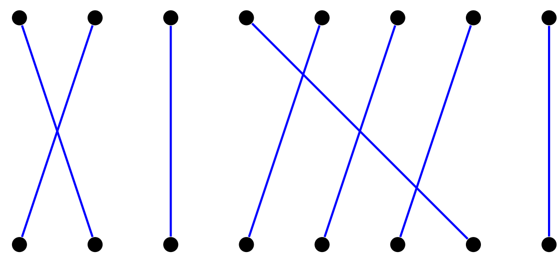


Figure 7. The perfect matching, M'

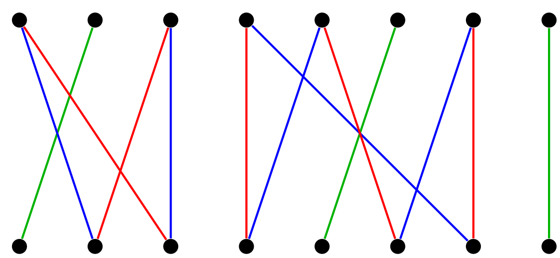


Figure 8

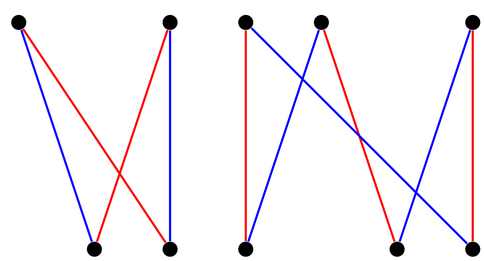


Figure 9. The perfect matching, $M \triangle M'$

If we select the left cycle as C, then we find that everything outside of C is a perfect matching. We don't care about whether the common edges stay or are removed. We would just follow the exact same matching. Now we have the right cycle, and we could choose what remains of M or M' to be the matching for the remaining vertices. Thus, all cycles are evenly-placed, and as a result of our assumption, properly-signed.

We have just seen that $M \triangle M'$ can have as many or as little cycles as possible. Let there be k cycles. Then, if they are all properly signed, for all $i \in k$, we know $|C_i| = 2l_i$ and $n_{C_i} \equiv l_i - 1 \pmod 2$. Thus, for a cycle, C_i, it contributes a factor of $(-1)^{n_{C_i}}$ since we only care about the negative edges. Thus, $\prod_{e \in C_i} \sigma(e) = (-1)^{(l_1-1)+\ldots+(l_k-1)}$. Now, we need to show that

$$sgn(\pi) = sgn(\pi') \cdot (-1)^{(l_1-1)+\ldots+(l_k-1)}.$$

We can do so by claiming that π and π' differ by L transpositions. To do so, we look at a cycle, l_i, and show that it takes $l_i - 1$ transpositions to change π to π'. To do so, at step t, we will find a j and k such that $\pi(j) = \pi'(k) = t$. Then, we will swap j and k in π. For example, take the right cycle from figure 10. We find that our permutation for π is (1,2,3) and the permutation for π' is (2,3,1). Theoretically, it should take 2 steps to get π to π'.

We start step 1. We find that $\pi(1) = \pi'(3) = 1$. We switch positions 1 and 3 in π. The permutation for π is now (3,2,1).

Now we start step 2. We find that $\pi(2) = \pi'(1) = 2$. Thus, we switch positions 1 and 2 in π to get a permutation of (2,3,1).

This will always work. For the all the steps except the last one, we find that we correct exactly one position. With step t, we have corrected at least 1 position. If we have $\pi(j) = \pi'(k) = t$ and then swap j and k in π, then we now have $\pi(k) = \pi'(k) = t$. Note that we cannot undo this correction since it would require us to go back to the t^{th} step.

However, what happens if we somehow make two corrections at once? That would mean that $\pi(j) = \pi'(k) = t$ and $\pi(k) = \pi'(j)$. This will never work since that would mean that there is another independent cycle. However, we have one exception. Every time we take a step, excluding the last one, we are making the length of the cycle shorter by 2. At the last step, we have a cycle with length 4, and as a result, we fix two positions at once. Thus, for a cycle, C_i, we know that it will take exactly $l_i - 2 + 1 = l_i - 1$ steps. Thus,

$$sgn(\pi) = sgn(\pi') \cdot (-1)^{(l_1-1)+\ldots+(l_k-1)}.$$

As a result, $sgn(M)sgn(M') = 1$ and our lemma has been proven. ■

Now, we aim to derive a method to easily find a Kasteleyn signing.

8.2. **Second Lemma.** We will start by first graphing a 2×3 grid with faces.

We start by numbering each square with a face, as shown in figure 10. Technically, you could have a face that includes more squares, but for the purposes of this theorem, we will include *exactly* one square. The purpose of these faces is so that we can use Euler's formula, our lemma, and many of our definitions.

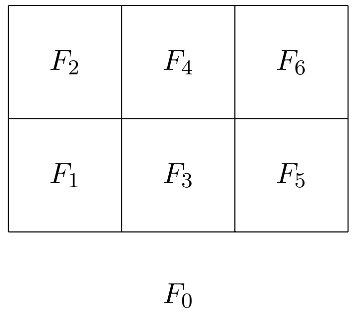


Figure 10. Our 2×3 grid with faces

Definition 8.3. Euler's polyhedron formula is $V + F = E + 2$ where V, F, and E are the number of vertices, faces, and edges, respectively.

A cycle will encompass one or more faces. If it encompasses more than one, then there should be some cycles within our chosen cycle, i.e. the cycles around *exactly* one face.

Lemma 8.4. *Fix a planar drawing of a bipartite, planar, 2-connected graph, G, with signing σ. If the boundary cycle of every inner face is properly signed, then σ is Kasteleyn.*

Proof. We will begin with Euler's formula by figuring out V, F, and E. We find that V is $r + 2l$ where r is the number of vertices inside of C. We find that $E = \frac{1}{2}(|C| + |C_1| + ... + |C_k|)$ since each edge appears on two cycles. Finally, F is $k+1$ since we include the outer face. Thus, plugging the terms into $V + F = E + 2$ gives us

$$r + 2l + k + 1 = \frac{1}{2}(|C| + |C_1| + ... + |C_k|) + 2$$

. We know that r is even since C is evenly-placed. We didn't assume that C is evenly-placed in the lemma, but it allows us to use the first lemma. Thus, by simplifying and reducing to mod 2, we get:

$$\begin{aligned} r + 2l + k + 1 &= \frac{1}{2}(|C| + |C_1| + ... + |C_k|) + 2 \\ \Rightarrow r + 2l + k + 1 &\equiv l + l_1 + ... + l_k + 2 \\ \Rightarrow r + 2l + k + 1 &\equiv l + l_1 + ... + l_k + 2 \pmod 2 \\ \Rightarrow l + k + 1 &\equiv l_1 + ... + l_k + 2 \pmod 2 \\ \Rightarrow l - 1 &\equiv l_1 + ... + l_k - k \pmod 2. \end{aligned}$$

As a result, we aim to prove that $l_1 + ... + l_k + k$ is the same parity as n_C.

We can do so by noting that each negative edge appears on exactly 2 cycles. Thus, $n_C + n_{C_1} + ... n_{C_k}$ is even. We can then find that n_C is the same parity as $n_{C_1} + n_{C_2} ... n_{C_k}$. We know that $C_1, C_2, ... C_k$ are properly signed since we assumed that they are. Note that this is not circular reasoning because these faces enclose around exactly one face while C could enclose around more. Since $C_1, C_2, ...C_k$ are properly signed, we know that $l_1 - 1 \equiv n_{C_1} \pmod 2$, $l_2 - 1 \equiv n_{C_2} \pmod 2$, ... $l_k \equiv n_{C_k} - 1 \pmod 2$. Thus, if we plug everything into our equation with n_C, then we get

$$n_C \equiv (l_1 - 1) + (l_2 - 1) + ... + (l_k - 1) \equiv l_1 + l_2 + ... + l_k - k \equiv l - 1 \pmod 2.$$

Thus, C is properly signed. ■

9. Results

So we have just shown that if we can create faces such that all the cycles around them are properly-signed, then we have a Kasteleyn signing. This is because any cycle is now properly-signed and because of lemma 8.2, we will get a Kasteleyn signing. Thus, we want to find a way such that all cycles around exactly one square is properly-signed.

Thankfully, this is easy to do. We can start by labeling each square with a face as shown in figure 11.

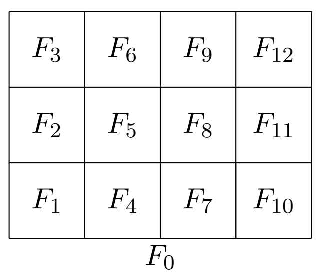


Figure 11. Our 3×4 grid with faces

Now, we can label all the edges with a 1. After that, we remove an edge that connects F_1 to F_0. For example, we remove the vertical line on the left. We are left with figure 12.

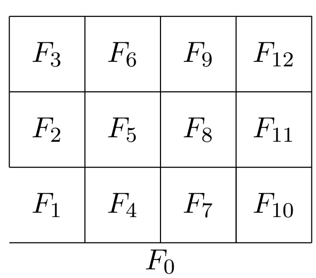


Figure 12

Afterwards, we can decide the weight of that edge to guarantee the cycle around F_1 is properly-signed. Similarly, as shown in figure 13, we can remove

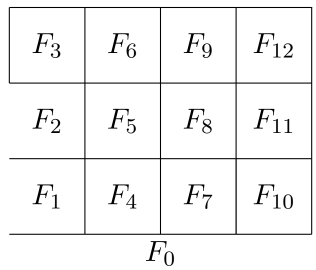


Figure 13

an edge that connects F_2 to F_0. It makes everything easier for us if we always choose an edge that is vertical.

We continue this process where we remove an edge so that it connects the face inside the square to the outer face, F_0. Eventually, figure 14 is our grid.

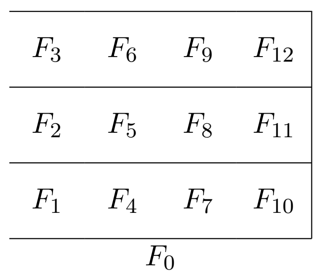


Figure 14

Now, we put back in the edges that we removed. However, we can change the weights to 1 or -1 to make each cycle around exactly one face properly-signed. For example, we now put back in the edge between F_7 and F_{10} as a -1.

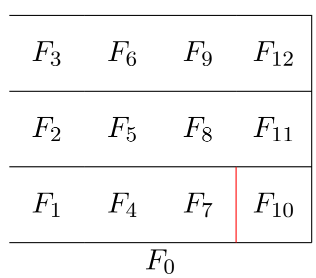


Figure 15

Similarly, we put two more red edges, which denote a -1.

When we put in the second row, we need positive 1's or blue edges.

In the end, we are just alternating between columns of 1 and -1.

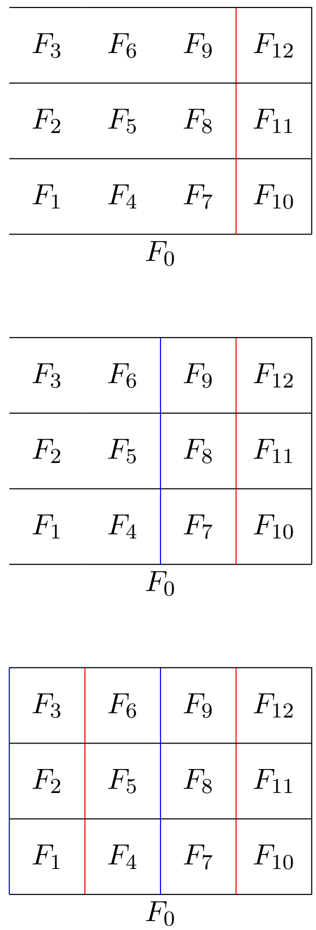


Figure 16. Our 3×4 grid with faces

Thus, our matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

with a determinant of 90.

10. Conclusion and Acknowledgements

This isn't the end though! There are other ways to solve this problem. Another form of the answer is $\prod_{k=1}^{m}\prod_{l=1}^{n}(4\cos^2\frac{k\pi}{m+1}+4\cos^2\frac{l\pi}{n+1})^{1/2}$. Additionally, a common example is to look at other shapes. We could remove certain squares. Can you still make a configuration? The Hall-Marriage Theorem is a great way to solve this question. Perfect matchings and bipartite graphs are still really useful. You could also consider how many ways there are to tile a grid with pieces that are shaped differently, perhaps Tetris tiles.

I would like to thank Lucas Fagan, my TA, and Dr. Simon Rubinstein-Salzedo, my instructor, for their patience and guidance. [1]

References

[1] Brendan W. Sullivan. How many ways can we tile a rectangular chessboard with dominos? URL: https://www.math.cmu.edu/~bwsulliv/domino-tilings.pdf, 2 2013.