An Introduction to Braid Groups

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Outline

Geometric Braids and Braid Diagrams

Artin Relations

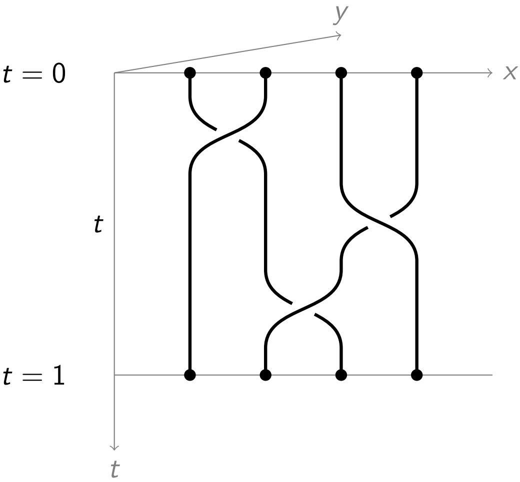
Fundamental Groups, Configuration Spaces, and Braids

Burau Representation

Geometric Braids

Definition

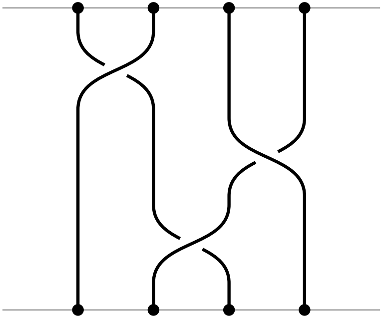
A **geometric braid** on n strands is a collection of n disjoint strands connecting n points on a plane to another n points on another parallel plane, with strands only moving vertically.



Braid Diagrams

Definition

A **braid diagram** is a 2D representation of a geometric braid where crossings are explicitly indicated. This is pretty much the same thing as the previous diagram, except that we only focus on the spots where crossings occur. Clearly, each geometric braid has an associated braid diagram.



When are two braids the same?

Imagine that you can pull these strands in a way which keeps all the strands at their starting and ending nodes, and that you don't tear any strands as you're doing this. Notice that doing this lets us intuitively see when two braids are essentially the same (isotopic).

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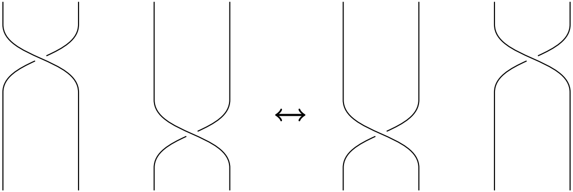
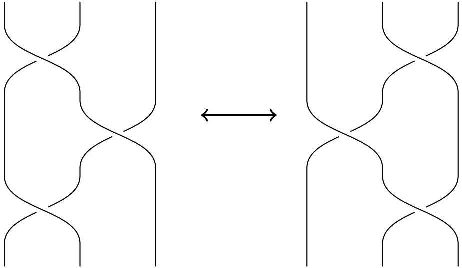
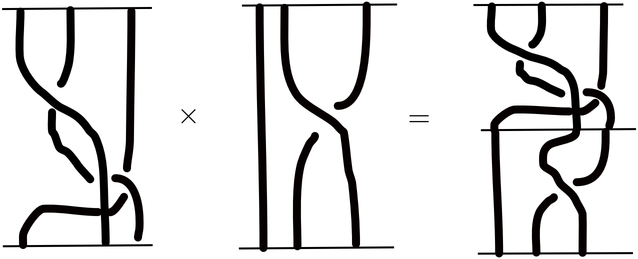


Another example:



Multiplication of braid diagrams

Using this visual intuition, we can also define the notion of composing two braids together.



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5. This implies that even though we started with just drawing overlapping strands, this simple visual construction has an underlying algebraic structure.

Artin Relations

Definition

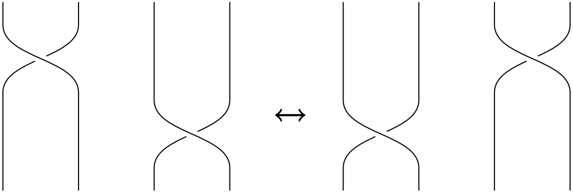
The **braid group** B_n on n strands is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the relations:

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| > 1$

What does this mean? Recall our previous diagrams...

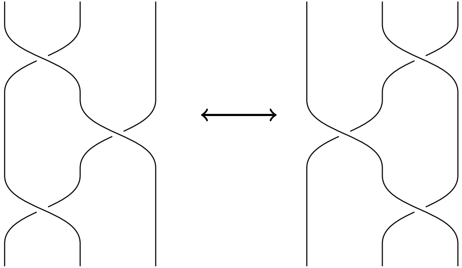
Why generators? How is this related to what we saw before?

Recall that the main feature of braid diagrams is that they specify which strands cross over each other. A generator captures the same information: some product of generators corresponds to a set of overlapping strands, and each generator σ_i represents the crossing of strand i over strand $i+1$. An inverse generator (σ_i^-) represents the i strand under the $i+1$ strand. We can now see why the Artin braid relations are true.



Commutativity relation:

$\sigma_1\sigma_3 = \sigma_3\sigma_1$



Skein relation:

$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

Fundamental Groups and Configuration Spaces

Definition

The **configuration space** $C_n(\mathbb{R}^2)$ is the space of n distinct points in $\mathbb{R}^2$.

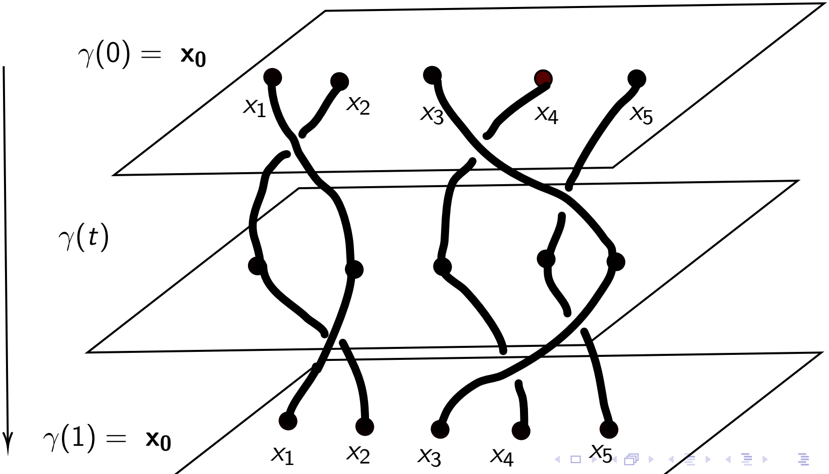
Theorem

The fundamental group $\pi_1(C_n(\mathbb{R}^2))$ *is isomorphic to the braid group* B_n*.*

We do not provide a proof of this theorem, but a quick sketch can give some intuition.

Connection to Braids

The *fundamental group* of a configuration space is the set of all loops in this space from some base point x_0, where two loops are considered the same if we can continuously deform one into the other over time.



Burau Representation

Definition

The **Burau representation** is a linear representation of the braid group B_n.

Theorem

The Burau representation maps B_n *to* $GL(n-1, \mathbb{Z}[t, t^{-1}])$.

Example

The representation of the generator σ_i is given by:

$$\rho(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

Why this representation?

Recall that we defined generators as being the "algebraic" versions of strands crossing over each other. If we consider individual strands of a braid as vectors, then the matrix encodes exactly this information: each strand from σ_1 to σ_{i-1} remains the same, and each strand from σ_{n-i-1} to σ_n remains unchanged, while i is crossed over $i+1$.

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Conclusion

- Geometric braids and braid diagrams provide a visual understanding.
- Braid groups are algebraically defined by Artin relations.
- There is a connection between braid groups, fundamental groups, and configuration spaces.
- The Burau representation provides a matrix representation of braid groups.

Thank you!