The Burau Representation of the Braid Group

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Abstract

This paper is an introduction to braid groups. We first introduce basic algebraic prerequisites, and continue by showing how braid groups arise in different contexts. This naturally leads into the discussion of representations, and we introduce the Burau representation of the braid group.

Acknowledgements

I would like to start by thanking Katherine Martin for being an excellent and very supportive guide to me throughout my work on this paper, particularly in clarifying topological notions I had never seen before. I would also like to thank Simon Rubinstein - Salzedo for giving me the opportunity to learn about braids so deeply. Lastly I would also like to thank Ms. Yvonne Secker, for being the first person who believed in my ability to understand mathematics, and for encouraging me to never give up on the pursuit of mathematical beauty.

0 Introduction

Braid groups are very rich mathematical objects which lend themselves to a variety of different interpretations. They are quite visually appealing and are easy to intuitively understand. Braid groups display a unique interplay between topology, geometry, and algebra, making them quite interesting structures for mathematicians from a diverse range of fields; braids can also be considered as an extension of the study of knots and links.

The study of braids can be dated all the way back to Gauss, who spent some time playing with them and trying to come up with rigorous ways of classifying separate braids. This was later picked up by Emil Artin in 1925, who first recognized the algebraic structure of braids. In 1962, Ralph Fox and Lee Neuwirth built on Artin's work and showed how braids can also be considered as the fundamental group of configuration spaces. The 1990 Fields medal was awarded to Sir Vaughn Jones for his work introducing the Jones polynomial, an algorithm for classifying knots and braids.

Although we do not delve into the applications of braids in this paper, they have also proved to be particularly useful outside of pure mathematics, especially in cryptography, molecular biology (it is easy to see how the interweaving strands of DNA lend themselves to this presentation), computer science (braids are used in the models of topological quantum computers) and, most notably, mathematical physics.

The connection between knots, braids, and physics is quite an old one, and has some history dating back to Lord Kelvin's models of the atom. Before the introduction of quantum mechanics at the beginning of the 20th century, physicists spent a lot of time proposing potential models of what matter might look like at its most fundamental scale. Lord Kelvin proposed an image of the atom as a set of interweaving "strings", thus imagining it, essentially, as a braid closed in on itself.

Although this theory was quickly disproved, braids once again appeared in theoretical physics at the end of the 20th century. Edward Witten's construction around quantum field theory used the theory of braids and knots to describe how our universe might behave at the smallest scale. This insight later led to

Witten's Fields medal. Quite surprisingly, braids also appeared in statistical mechanics, where Artin's canonical presentation corresponded to the Yang-Baxter equation.

However, we do not need to understand advanced mathematics, physics, or biology to recognise the simple beauty of braid groups. I hope that this introduction allows the reader to see for themselves why the subject is worth studying, and how the simple idea of weaving strands holds a lot of emergent beauty. As we will see, braids provide an elegant bridge between the worlds of algebra and topology.

We begin by introducing basic notions from group theory and representation theory in section 1. Section 2 explains Artin's algebraic presentation of braid groups and defines the braid relations, while providing some visual intuition. We consider the drawbacks of Artin's presentation in classifying braids. This leads to the geometric and topological views of braids in section 3, where we introduce the notion of *isotopies* to rigorously define equivalence between two braids. Section 4 presents braids as the fundamental group of configuration spaces. This lets us combine the algebra and geometry of braids in section 5, motivating the Burau representation.

1 Prerequisites

Before defining braid groups, it is important to cover a few basics.

1.1 Group Theory and Representations

Definition 1.1. A group G is a set endowed with a binary operation, $*$, which has the following properties:

- It is closed under composition.
- For every element x in G, there exists a unique inverse x^{-1}.
- The set G contains an identity element, e, such that $x * e = x$.

Note that groups are not necessarily commutative. Commutative groups are called *abelian*.

Definition 1.2. A *finitely generated group* is defined as a group G which consists of some finite set S of generators $\sigma_1, \sigma_2, \dots, \sigma_n$, such that any element of G can be expressed as a composition of the generators. Intuitively, this means we have a small set of elements which can generate any element in the group.

Definition 1.3. A *homomorphism* can be thought of as a map between elements of different groups. We define a homomorphism between two groups G and H as some mapping $\phi : G \to H$ such that for any elements $x, y \in G$, $\phi(x) * \phi(y) = \phi(x*y)$. If this mapping is bijective, it is called an *isomorphism*. This means that the two groups we are considering are essentially the same, the main difference between them being how we label individual elements.

Definition 1.4. A *representation* of a group G is defined as a homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the group of invertible matrices (or linear transformations) over a vector space V. This means that for all $g, h \in G$, $\rho(g)\rho(h) = \rho(gh)$. Intuitively, a representation maps each element of the group to an invertible matrix in such a way that the group operation is preserved. If the representation is injective (mapping separate elements in the group to separate matrices), the representation is called faithful. Representations allow us to more simply understand groups which, if only considered algebraically, are difficult to work with.

We are now ready to see how braids can be considered from an algebraic perspective.

2 The Artin braid group and braid relations

Before going into rigorous presentations, it's useful to get some visual intuition into what exactly braids are. We will precisely define everything after intuitions are presented. The figure below is a basic example of some braids.

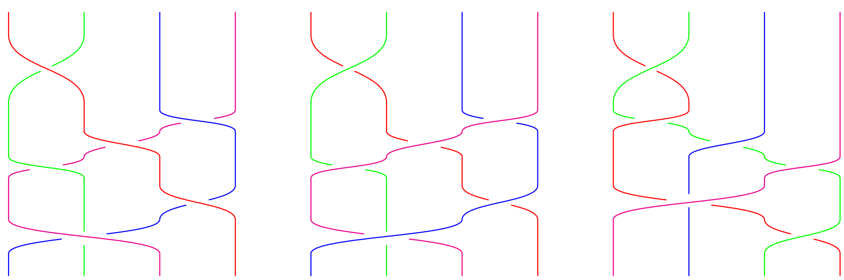

Figure 1: A few braids.

Definition 2.1. The *Artin braid group* B_n is a finitely generated group with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ satisfying the following relations:

1. The commutativity relation:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| > 1 \tag{1}$$

2. The Skein relation:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{2}$$

   for all i.

3. The identity relation:

$$\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1 \tag{3}$$

Here, σ_i represents the i-th generator of the braid group, corresponding to a crossing between the i-th and $(i+1)$-th strands.

These relations looks confusing and without motivation at first, but they are better understood with some visual intuition. The commutativity relation essentially shows how if the only difference between two braids is the height at which crossings occur, they can be considered the same braid:

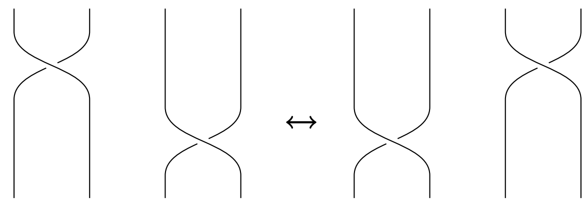

Figure 2: $\sigma_1\sigma_3 = \sigma_3\sigma_1$

The picture above shows two different braids, each with four strands. From the braid on the left, we can imagine pulling the first crossing on the left down, and the second crossing on the right up. The generators in the braid group correspond to how a given braid crosses over another one; for example, in the braid above, the first strand crosses over the second strand, and the third strand crosses over the fourth strand, so this is denoted as $\sigma_1\sigma_3$. These two braids are essentially the same, the only difference between them being the height at which the crossings occur. This clarifies the commutativity relation.

The second relation relates to another visual intuition, explaining when two braids can be considered the same:

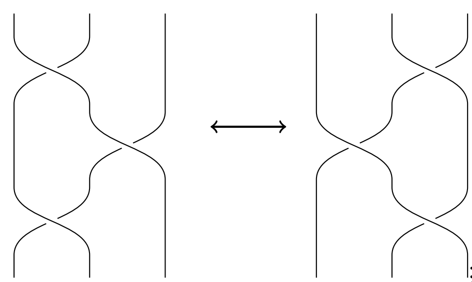

Figure 3: $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

In both of these braids, the third strand is underneath all the other strands, and the first strand is above both other strands. If the second strand on the right is "pulled" to the left and the third strand is pulled to the bottom, the two braids will look identical.

These relations are referred to as the *braid relations*, and any element of B_n is referred to as a braid. B_n is the set of all braid with n strands.

The algebraic view of braids also lets us see the natural homomorphism from braid groups to the symmetric group. To see this, we prove the following lemma:

Lemma 2.2. *Let $f : B_n \to G$ be a group homomorphism from B_n to a group G. Then the elements $\{a_i = f(\sigma_i)\}_{i=1}^{n-1}$ satisfy the braid relations. Conversely, if some $\{a_1, \dots, a_{n-1}\} \in G$ satisfy the braid relations, then there exists a unique group homomorphism $f : B_n \to G$ such that $a_i = f(\sigma_i)$ for any $i \in \{1, \dots, n-1\}$.*

Proof. The first direction only requires a quick verification. Given $f : B_n \to G$ a group homomorphism and $i, j \in \{1, \dots, n-1\}$, we have

$$a_i a_j = f(\sigma_i)f(\sigma_j) = f(\sigma_i\sigma_j) = f(\sigma_j\sigma_i) = f(\sigma_j)f(\sigma_i) = a_j a_i,$$

and for $i \in \{1, \dots, n-2\}$ we have

$$a_i a_{i+1} a_i = f(\sigma_i)f(\sigma_{i+1})f(\sigma_i) = f(\sigma_i\sigma_{i+1}\sigma_i) = f(\sigma_{i+1}\sigma_i\sigma_{i+1}) = f(\sigma_{i+1})f(\sigma_i)f(\sigma_{i+1}) = a_{i+1}a_i a_{i+1}.$$

For the other direction, let F_n denote the free group generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$. Suppose $\{a_1, \dots, a_{n-1}\} \in G$ satisfy the braid relations. Then there exists a unique group homomorphism $\tilde{f} : F_n \to G$ such that $\tilde{f}(\sigma_i) = a_i$ for all $i \in \{1, \dots, n-1\}$. Note because $\{a_i\}_{i=1}^{n-1}$ satisfy the braid relations,

$$\tilde{f}(\sigma_i\sigma_j) = \tilde{f}(\sigma_i)\tilde{f}(\sigma_j) = a_i a_j = a_j a_i = \tilde{f}(\sigma_j)\tilde{f}(\sigma_i) = \tilde{f}(\sigma_j\sigma_i)$$

for all $i, j \in \{1, \dots, n-1\}$. Moreover, for any $i \in \{1, \dots, n-2\}$

$$\tilde{f}(\sigma_i\sigma_{i+1}\sigma_i) = a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = \tilde{f}(\sigma_{i+1}\sigma_i\sigma_{i+1}).$$

Thus $\tilde{f}$ induces a unique group homomorphism $f : B_n \to G$ such that $a_i = f(\sigma_i)$ for all $i \in \{1, \dots, n-1\}$.

□

We can apply this lemma to project to the symmetric group $G = \mathfrak{S}_n$. Recall that $\mathfrak{S}_n$ is defined as all the permutations of the set $\{1, 2, \dots, n\}$. We consider the *simple transpositions* $s_1, s_2, \dots, s_{n-1} \in \mathfrak{S}_n$, where s_i only permutes i and $i+1$ while leaving all the other elements of $\{1, 2, \dots, n\}$ unchanged. Clearly, the simple transpositions satisfy the braid relations: therefore, the previous lemma implies that there exists a unique homomorphism $f : B_n \to \mathfrak{S}_n$, such that $s_i = f(\sigma_i)$ for all $i = 1, 2, \dots, n-1$. It is also straightforward to see that the simple transpositions generate $\mathfrak{S}_n$ (if we keep switching things around one by one in any order, we will eventually get all possible switches of the symmetric group). Therefore, this homomorphism must be surjective. This connection between braid groups and the symmetric group will be important to keep in mind in later sections.

3 Geometric braids and isotopy

In the previous section, we appealed to diagrams in order to make sense of the algebraic structure of braid groups. We can make this visual intuition more rigorous by defining geometric braids in terms of homeomorphisms.

Definition 3.1. A function $f : X \to Y$ between two topological spaces X and Y is called a *homeomorphism* if it satisfies the following conditions:

1. f is a *bijection*, meaning that it is both injective (one-to-one) and surjective (onto).
2. f is *continuous*, meaning that for every open set $U \subseteq Y$, the preimage $f^{-1}(U) \subseteq X$ is also open.
3. The inverse function $f^{-1} : Y \to X$ is also *continuous*, meaning that for every open set $V \subseteq X$, the preimage $(f^{-1})^{-1}(V) = f(V) \subseteq Y$ is open.

If such a function f exists, we say that X and Y are *homeomorphic*, and the function f is called a *homeomorphism*.

Definition 3.2. A *geometric braid on n strings* is a subset $\beta \subset \mathbb{R}^2 \times [0,1]$ consisting of n disjoint topological intervals (called the *strings of β*) such that the projection

$$\mathbb{R}^2 \times [0,1] \to [0,1]$$

maps each string homeomorphically onto $[0,1]$. Further, β satisfies:

1. $\beta \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), \ldots, (n,0,0)\}$
2. $\beta \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1), (2,0,1), \ldots, (n,0,1)\}$

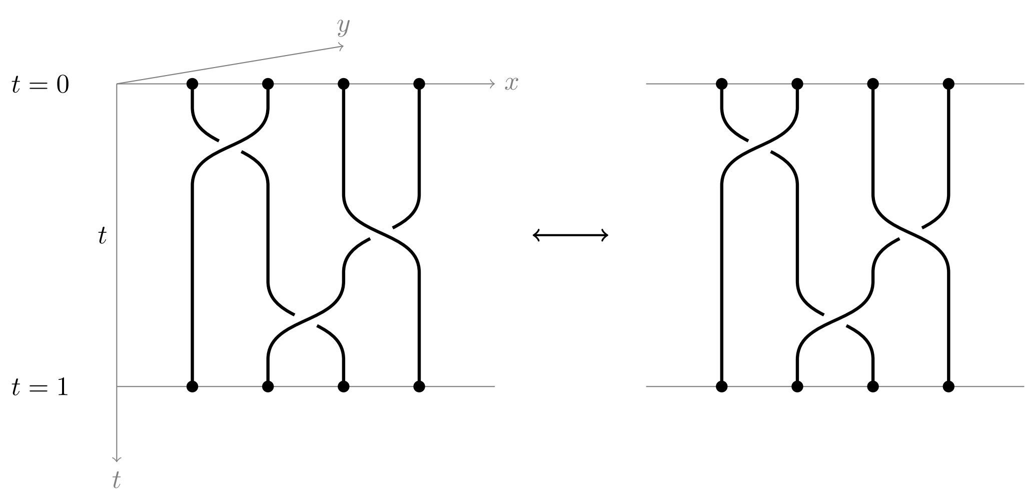

Figure 4: A geometric braid and its associated braid diagram.

Each string of β starts from some point $(i,0,0)$ and ends at some $(s(i),0,1)$ where $i, s(i) \in \{1,2,\ldots,n\}$. By looking at where each string ends up, we obtain a permutation $(s(1), s(2), \ldots, s(n))$ of the set $\{1, 2, \ldots, n\}$, referred to as the *underlying permutation of* β.

The underlying permutation of the braid in figure 4 is (3, 1, 4, 2).

An example of a geometric braid is displayed on the left of figure 3. Given some geometric braid, we can project a geometric braid onto $\mathbb{R} \times [0, 1]$ and note where individual strands intersect. This gives the braid diagram on the right, which is exactly what we used for visual intuition around the braid relations in the previous section. Clearly, any kind of braid diagram d corresponds to some geometric braid β.

Having now rigorously established the geometric notion of braids, it is natural to question when two braids can be considered the same. Simply representing braids through generators could make this difficult, since we get a long line of generators we need to work through using the relations in order to show equality.

The geometric perspective on braids simplifies this, since we can visualise when two braids are equal to each other by a series of transformations. The notion of "braid equality" is rigorously defined through by *isotopy*:

Definition 3.3. We define two geometric braids β_1, β_2 on n strands as *isotopic* to each other if there exists a continuous map

$$F : \beta_1 \times [0, 1] \to \mathbb{R}^2 \times [0, 1]$$

such that:

$$F(x, 0) = \mathrm{id}_{\beta_1}$$
$$F(x, 1) = \mathrm{id}_{\beta_2},$$

and at any point $t \in [0, 1]$, $F_t : \beta_1 \to \mathbb{R}^2 \times [0, 1]$ defines a geometric braid.

Intuitively, this definition tells us that two braids are isotopic, or "the same", if we can continuously deform one of the braids into the other in a way which preserves the geometric braid structure throughout the entire transformation. It is important to note that isotopy actually has a wider definition outside of braids which is not directly relevant to this paper and is a wider idea in algebraic topology. We do not delve further into this, since a basic understanding of isotopy in the context of braids is enough for us to understand what is meant by "equality" between braids. Interested readers should consult X. As the parameter t varies from 0 to 1, we are, essentially, pulling the strands of the braid in such a way that none of the strands get torn apart or removed from their end points. This parameter is introduced to formalise the idea that an isotopy is a deformation over time; an intuitive way to think about it is that as t varies from 0 to 1, we are continuously deforming one braid into another. This is why at $t = 0$ we have the braid β_1 unchanged, and at $t = 1$ we have the braid β_2 unchanged. If such a transformation exists between two braids, they can be considered as the same braid. Clearly, two braid diagrams d_1, d_2 are isotopic if and only if their corresponding geometric braids β_1, β_2 are isotopic. For example, the three braid diagrams below are isotopic:

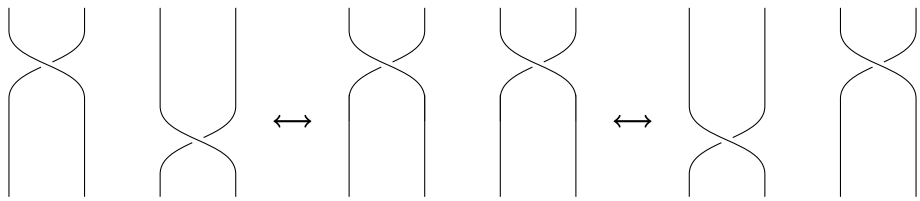

Figure 5: Isotopic braids

We can also define multiplication on two geometric braids. Given two braid diagrams d_1, d_2, the product $d_1 d_2$ is obtained by placing d_1 on top of d_2 and compressing the result to fit into $\mathbb{R} \times [0, 1]$.

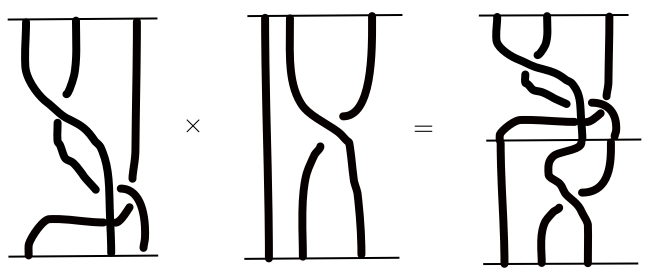

Figure 6: Multiplication of braids. A braid with n straight strands is the identity element.

Having defined multiplication on geometric braids, we can now link together the algebraic and geometric perspectives. Let $\mathcal{B}_n$ be the set of all geometric braids with n strands. We have already shown that $\mathcal{B}_n$ is closed under multiplication, since putting together any two braids gives another braid. We have also defined the identity element of $\mathcal{B}_n$. By showing that each $\beta \in \mathcal{B}_n$ has a unique two sided inverse, we prove that $\mathcal{B}_n$ is, in fact, a group.

Lemma 3.4. *Every geometric braid $\beta \in \mathcal{B}_n$ has a unique two sided inverse $\beta^{-1} \in \mathcal{B}_n$.*

Proof. We start by defining two braids σ_i^+ and σ_i^- for each i $\in \{1, 2, \ldots, n\}$. σ_i^+ is a braid with only a single crossing, at which the i-th strand crosses over the $i+1$-th strand. σ_i^- is a braid with the same crossing, but the i-th strand is instead going under the $i+1$-th strand. Firstly, note that $\sigma_i^+ \sigma_i^- = \sigma_i^- \sigma_i^+ = 1$.

We claim that $\sigma_1^+, \ldots, \sigma_{n-1}^+, \sigma_1^-, \ldots, \sigma_{n-1}^-$ generate $\mathcal{B}_n$. Consider a braid β on n strings and its associated braid diagram d. By slightly deforming the region around crossings in β, d becomes isotopic to some braid diagram d' in which the heights of crossings are all different. More rigorously, if d' has crossings at points

$(x_1, s_1), \ldots, (x_k, s_k) \in \mathbb{R} \times [0, 1]$, then the second coordinates $s_1, \ldots, s_k \in [0, 1]$ are all distinct from each other: Then we can choose some $t_0, \ldots, t_k \in [0, 1]$ such that

$$0 = t_0 < t_1 < \ldots < t_k = 1$$

so the intersection $d \cap (R \times [t_j, t_{j+1}])$ for $j \in \{0, \ldots, k-1\}$ contains only one crossing. Then each such intersection will either be a diagram of σ_i^+ or σ_i^-: Having decomposed our original braid into separate σ_i, we can now write d' as a product:

$$d' = \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \cdots \sigma_{i_k}^{\varepsilon_k}$$

where each ε_i is either $+$ or $-$ and $i_1, \ldots, i_k \in \{1, 2, \ldots, n-1\}$. As we've previously shown, since d' is isotopic to d, then its associated geometric braid, β', must be isotopic to β. We now define a new braid diagram, d'^{-1}, such that

$$d'^{-1} = \sigma_{i_1}^{-\varepsilon_1} \sigma_{i_2}^{-\varepsilon_2} \cdots \sigma_{i_k}^{-\varepsilon_k}$$

We have previously established that $\sigma_i^+ \sigma_i^- = \sigma_i^- \sigma_i^+ = 1$, so the geometric braid β^{-1} associated with d'^{-1} must be a two sided inverse of β', and hence also of $\beta \in \mathcal{B}_n$. □

Therefore, $\mathcal{B}_n$ forms a group. The most important aspect of this is that this group is isomorphic to the Artin braid group we defined in section 2:

Theorem 3.5. *Let $\varepsilon = \pm 1$. Then there exists a unique isomorphism $\phi_\varepsilon : \mathcal{B}_n \to B_n$.*

The proof of this is quite long, not particularly interesting and involves introducing the machinery of Reidemester moves. We do not prove this theorem in this paper. For a proof, see [KT08]. However, we already motivated the first algebraic definitions with some visuals, so this isomorphism should seem intuitively true.

Having presented the algebraic and geometric picture of braids, we can now move on to the topological view of braids.

4 Braids and configuration spaces

One of the most appealing views on braids comes from considering the fundamental group of a configuration space. We first define a few necessary terms.

Definition 4.1. For a given topological space M, the *configuration space* of n distinct points in M is defined as the space of all possible ordered n-tuples of distinct points in M. This is denoted as $\mathrm{Conf}_n(M)$:

$$\mathrm{Conf}_n(M) = \{(x_1, x_2, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

Definition 4.2. Let X and Y be sets and $f, g : X \to Y$ be continuous. Then, f and g are *homotopic* if there exists a continuous function $F : X \times [0, 1] \to Y$ such

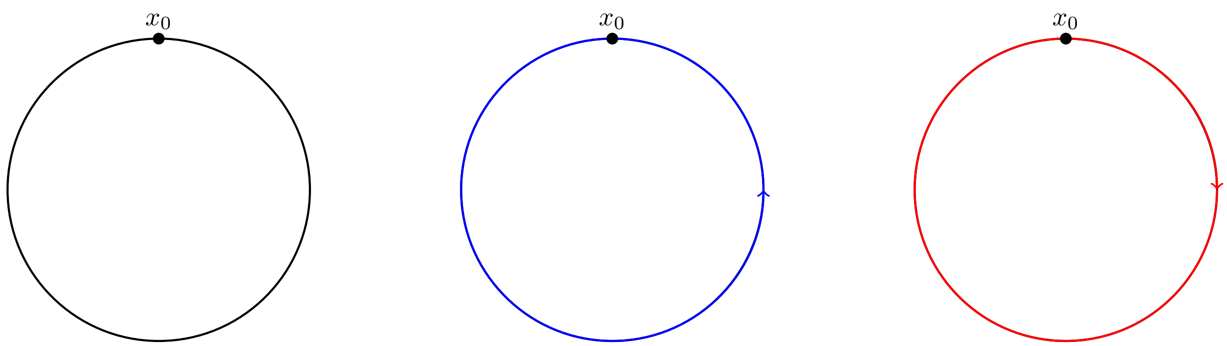

Figure 7: Elements of a circle's fundamental group.

that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. F is called a homotopy between f and g, and we write $f \sim g$. This is a generalisation of the notion of isotopy we had previously used to define equivalence classes of braids. This definition is quite similar to the previous one: a homotopy can be considered as a continuous deformation over time.

Definition 4.3. The fundamental group $\pi_1(X, x_0)$ of a topological space X with basepoint x_0 is the group of homotopy classes of loops based at x_0. Intuitively, the fundamental group of a space is the set of different loops that can be drawn in some space X starting from x_0. The homotopy of two loops refers to loops which can be continuously deformed into each other without lifting a given loop off of the space or breaking it. This definition is quite confusing, and it helps to have some examples.

Consider the fundamental group of a circle. We can imagine a loop starting from the bottom of the circle which goes around the circumference zero, once, twice, three times, or any integer number of times:

The diagram above shows three different elements of the fundamental group of the circle, and also makes it a bit clearer as to why this set of loops starting from a point is a group. We define the identity element of a fundamental group as a loop which doesn't wind around at all, so does nothing. We can also imagine concatenating loops by putting the end of one loop on the beginning of another. The red and blue circles also show loops going in opposite directions, which shows how each loop can have an inverse. We do not rigorously prove that the fundamental group is, in fact, a group, but this is hopefully clear from the diagram. Since a circle can have any integer number of loops around it, the fundamental group of the circle is simply $\mathbb{Z}$. This can be contrasted with the fundamental group of the torus. Since the torus has two axis along which loops can be drawn, its fundamental group is $\mathbb{Z} \times \mathbb{Z}$.

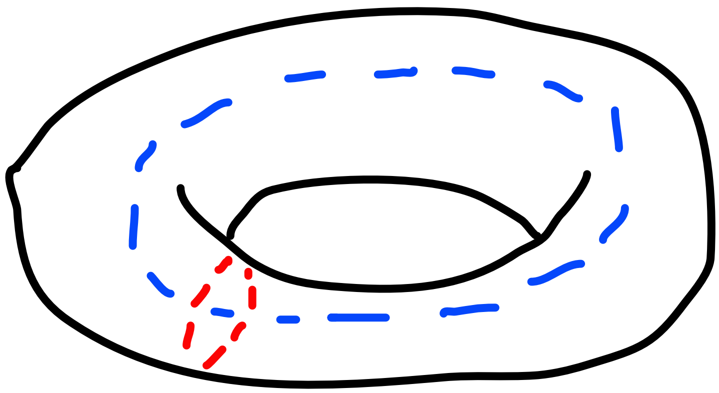

Figure 8: An element of the torus' fundamental group.

Having defined configuration spaces and fundamental groups, we can now link these concepts together to see how the braid group arises.

Recall that the fundamental group is a set of loops which all have equivalence up to homotopy. Therefore, the fundamental group of some configuration space, for example, $\mathrm{Conf}_n(\mathbb{C})$, is the set of paths from some point $\boldsymbol{x_0} \in \mathrm{Conf}_n(\mathbb{C})$ back to itself. Given such a based loop $\gamma : [0,1] \to \mathrm{Conf}_n(\mathbb{C})$ at each time $t \in [0,1]$, $\gamma(t)$ gives a new ordered configuration of points on the plane. We can think of this as the trajectories $\gamma_i : [0,1] \to \mathbb{C}$ of n points moving in the plane $\mathbb{C}$, given that the points do not collide and end up back where they started, which is the ordered configuration $\boldsymbol{x_0} = (x_1, x_2, \ldots, x_n)$.

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_k(t)) \text{ for } t \in [0,1], \text{ with } \gamma(0) = \gamma(1) = \boldsymbol{x_0}.$$

Hence one can easily visualize the fundamental group of a configuration space as a set of interweaving strands; that is, a braid.

Notice that in the diagram below, each strand of the braid begins and ends at the same node. Braids which have this property are called *pure*, and their set is called the *pure braid group*. This notion of using configuration spaces and the fundamental group to describe braids can be generalised outside of just pure braids. It is easy to see that the symmetric group $\mathfrak{S}_n$ acts on $\mathrm{Conf}_n(\mathbb{C})$ by permuting the coordinates. By modding out the configuration space by the symmetric group, we obtain the *unordered configuration space*:

$$\mathrm{UConf}_n(\mathbb{C}) = \mathrm{Conf}_n(\mathbb{C})/\mathfrak{S}_n$$

Applying the same reasoning as above, it is easy to see that the fundamental group of an unordered configuration space is the braid group.

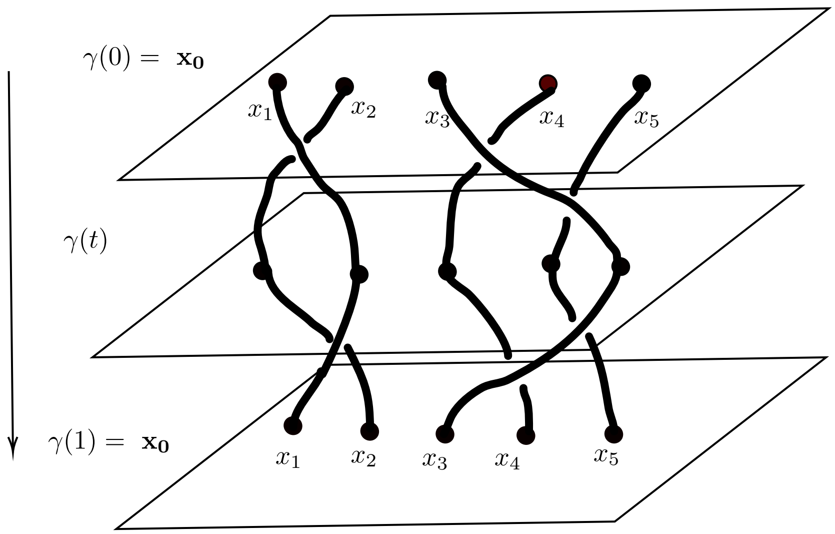

Figure 9: A braid as the fundamental group of a configuration space.

5 The Burau representation

Having now seen how braids can arise in different contexts, we introduce the Burau representation of the braid group; that is, an explicit homomorphism from the braid group B_n to the group of $n \times n$ invertible linear matrices.

Definition 5.1. A *Laurent polynomial* in t with coefficients in the field $\mathbb{Z}$ is a sum of the form

$$\lambda = \sum_{k \in \mathbb{Z}} n_k t^k$$

where $n_k \in \mathbb{Z}$. Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ denote the ring of such Laurent polynomials.

The Burau representation is a linear representation of B_n consisting of $n \times n$ matrices over Λ.

We have previously seen that in all of our perspectives on braids, the structure of a given braid can be accurately captured by specifying where different crossings occur. Keeping this in mind, we can see why braids have the following representation:

Definition 5.2. Let $n \geq 2$. Define U_i as the following $n \times n$ matrix with entries in the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$:

$$U_i = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix}$$

where I_k denotes the $k \times k$ identity matrix.

Notice each U_i has a block diagonal form: the blocks are $(i-1)\times(i-1)$ and $(n-i-1)\times(n-i-1)$ identity matrices, and the 2×2 matrix

$$U = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}.$$

The matrix U is key in understanding why exactly braid groups are represented like this. If $i = 1$, there is no identity matrix in the upper left corner; for each given i, U_i keeps all vectors from 1 to $i-1$ unchanged, permutes the positions of vectors $i, i+1$, and leaves all the other vectors until n also unchanged. Effectively, this matrix encodes the "crossing over" action we had previously defined using generators! As t varies, the crossing occurs "over time". Considering individual strands of a braid as vectors, the matrix represents crossings.

Although this makes the matrix representation itself intuitively clear, we still need to show that this representation is always invertible.

Lemma 5.3. *Each $U_i, i \in \{1, \ldots, n-1\}$, is invertible.*

Proof. The Cayley-Hamilton theorem states that any 2×2 matrix M satisfies the equation

$$M^2 - \mathrm{tr}(M)M + \det(M)I_2 = 0.$$

Thus, U satisfies $U^2 - (1-t)U - tI_2 = 0$. The identity matrices also satisfy this relation, so we get that for all $i \in \{1, \ldots, n\}$,

$$U_i^2 - (1-t)U_i - tI_n = 0.$$

This is equivalent to

$$U_i(U_i - (1-t)I_n) = tI_n.$$

Multiplying by t^{-1} on both sides shows that U_i is invertible (over Λ) and that

$$U_i^{-1} = t^{-1}(U_i - (1-t)I_n) = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t^{-1} & 1-t^{-1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix}.$$

□

Naturally, we expect this representation to satisfy the braid relations.

Lemma 5.4. *Let $n \geq 2$. The matrices U_i for all $i \in \{1, \ldots, n\}$ satisfy the braid relations, i.e.*

$$U_iU_j = U_jU_i \quad \text{for all } i, j \text{ with } |i-j| \geq 2$$

$$U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1} \quad \text{for all } i \in \{1, \ldots, n-2\}.$$

Proof. This is fairly straightforward to prove, since all of the relations can be proven by performing some matrix multiplication. For the first relation, consider $i, j \in \{1, \ldots, -1\}$ such that $|i - j| \geq 2$. Multiplying U_i, U_j we see that

$$U_iU_j = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix} \begin{bmatrix} I_{j-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}$$

$$= \begin{bmatrix} I_{i-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{j-i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-t & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{n-j-2} \end{bmatrix}$$

$$= \begin{bmatrix} I_{j-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-j-1} \end{bmatrix} \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix} = U_jU_i.$$

Hence the braid commutavitity relation is satisfied. We do the same thing for the second relation; we just need to show that $U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$ for $i = 1, \ldots, n-2$. So we need to verify the equality

$$\begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 1 \end{bmatrix}.$$

Multiplying out terms on the left side gives:

$$\begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-t & t-t^2 & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

And the right side gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-t & t-t^2 & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

So the representation obeys the braid relations.

□

By the result of lemma 2.2, there exists a group homomorphism

$$\psi_n : B_n \to \mathrm{GL}_n(\Lambda)$$

such that $\psi_n(\sigma_i) = U_i$ for every $i \in \{1, \ldots, n-1\}$. This aligns with our previous intuition around generators corresponding to matrices. This is the *Burau representation* of B_n. Another useful intuition around understanding this representation comes from Sir Vaughn Jones: imagine the separate strands of a braid each corresponding to a bowling alley. Once a bowling ball is rolled on a given strand and approaches a crossing, it will either stay on its original strand with probability $1-t$, or it will fall onto the strand below with probability t. The Burau representation is known to be faithful for $n = 3$, and unfaithful for $n \geq 5$. For a proof of this result, see [KT08]. The faithfulness of the Burau representation for $n = 4$ remains an open problem as of the writing of this paper.

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